A SOLUTION OF NONLINEAR PLANE STRAIN PROBLEMS IN DYNAMIC SOIL MECHANICS

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PREFACE

This study is part of a project sponsored under Grant NsG-604 to The University of Texas at Austin from the National Aeronautics and Space Administration, Langley Research Center, Hampton, Virginia. It is the tenth in a series of reports issued at The University of Texas at Austin on the interaction between soils and geometrical models under dynamic and static loading.

The computer program included in this report was written in Fortran IV Language for the CDC 6600 computer. With minor changes the program would be compatible with an IBM 7090 computer.

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ABSTRACT

A theoretical study was performed to define the response of a cohesionless sand of medium density to different rates of loading. The deformation properties were assumed to be represented by two nonlinear curves representing axial strain vs. the deformation modulus and axial strain vs. the lateral strain ratio. A theoretical analysis was performed to support this assumption. The general case of finite deformation was considered.

The problem investigated was the penetration of a rigid plate into a vertical surface bounded by a horizontal surface. The force deformation histories under different rates of loading were obtained, as well as the stress distribution in the soil mass. An empirical formula, based on theoretical results, was suggested to relate the ultimate load to the rate of loading. An elasto-plastic analysis was also suggested.

An iterative process using the point relaxation technique was utilized to solve the nonlinear equations. A computer program was written for that purpose.

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SYMBOLS AND NOTATIONS

a_0, a_1 through a_6	Constants of the axial strain vs. modulus of deformation polynomial
a_1, a_2, a_3	Used again in Appendix A as coordinates of a point in the undeformed state
a_{ik}	Partial differentiation of a_i with respect to the kth independent variable
A	Area of the plate
A, B, C, D, F, L, M, N, PH, PS, Q, R	Symbolic parameters for partial derivatives
AA, AAM, AM, BB, BK, CC, DD, DN, DNN, EE, FF, FK, FN, GG, LL, MM	Symbolic parameters written in terms of time and displacements
C	Used again in Chapter V as an integration parameter
d	Movement of the plate
DX	Symbolic parameter written in terms of displacements
E	Elastic Young's Modulus
E'	Generalized modulus of deformation
F_0, F_1, F_2, F_1', F_2'	Constants or functions of the strain invariants
F_x	Body force in the x direction
F_y	Body force in the y direction
G	Deformation index defined in terms of E' and ν'. Equivalent to the shear modulus G in the theory of elasticity
HT	Time increment in seconds
HX, HY	Increment lengths in the x and y directions in inches
i, j	Subscripts denoting the column and row number for location of the material position of a nodal point in the relaxation net
J_1, J_2, J_3	Invariants of strain at a point

K_1, K_2	Constants or functions of strain invariants
k	Subscript denoting time station
K	Constant in the yield function. Used again in Chapter V to denote the Average Earth Pressure Modulus.
P	Load carried by the plate due to a particular plate movement and for a particular rate of loading
PX, PY	Inertia force in the x and y direction
P_{ult}	Ultimate load carried by the plate for a particular rate of loading
$P_{ult_{max}}$	Maximum ultimate load carried by the plate based on P_{ult} for several rates of loading
R	Used in Chapter IV as a symbolic parameter. Used in Chapter V for the rate of loading.
REX, REY	Residual forces in the x and y directions
S_{ij}	Deviatoric component of stress
t	Time
T	Time elapsed since start of deformation
u, v	Normal displacements in the x and y directions
$\ddot{u}$, $\ddot{v}$	The second partial derivative of u and v with respect to time
u_x, u_y	The first partial derivatives of u with respect to x and y
u_{xx}, v_{xx}	The second partial derivative of u and v with respect to x
u_{xy}, v_{xy}	The second partial derivatives of u and v with respect to x and y
u_{yy}, v_{yy}	The second partial derivative of u and v with respect to y
U_1, U_2	Displacements in the X_1, X_2 directions as used in Appendix A
U_{jk}	The derivative of U_j with respect to the kth independent variable

v_y, v_x	The first partial derivative of v with respect to y and x
VDR, VSR, SHM, PHO	Stand for $(\lambda+2G)$, $(\lambda+G)$, G and ρ as used in the computer program
W_0, W_1, W_2, W_3, W_0' W_1, W_2', W_3', S, S'	Convergence parameters
x, y, z	Cartesion coordinates
x_i	Coordinate of a point in the deformed state as used in Appenidx A
x_{ij}	The derivative of x_i with respect to the jth independent variable
X, Y	Distances of a nodal point from origin of coordinates
X_1, X_2	Denote Cartesion coordinates in Appendix A
Z	Dimensionless parameter
α, β, $\mathcal{K}$	Constants or functions of strain invariants
γ_1, γ_2, γ_3, γ_4	Dimensionless physical constants
γ_s	Unit weight (lb/cu ft)
δ	Kronecker delta
ϵ_{ij}	Strain tensor
ϵ_x, ϵ_y	Normal strains in the x and y direction
ϵ_{xy}	Shear strain in the x - y plane
ϵ'_{ij}	Deviatoric component of strain
$\dot{\epsilon}_{ij}$	Derivative of strain with respect to time
λ	Deformation index defined in terms of E' and ν'. Equivalent to Lame' constant λ in the theory of elasticity
ν	Dimensionless physical constants
ν'	Generalized lateral strain ratio
ρ	Mass density (lb sec^2/in.^{4})
σ	Isotropic component of stress

σ_1, σ_2, σ_3	Principal stresses at a point
σ_x, σ_y, σ_z	Normal stresses in the x, y, and z directions
σ_{ij}	Stress tensor
σ_{xx}, σ_{yy}	Normal stresses in the x and y directions as they appear in the equilibrium equations
$\sigma_{xx,x}$	The first partial derivative of σ_{xx} with respect to x
σ_{xy}	Shear stress in the x - y plane
$\sigma_{xy,y}$, $\sigma_{xy,x}$	The first partial derivatives of σ_{xy} with respect to y and x
$\sigma_{yy,y}$	The first partial derivative of σ_{xy} with respect to y
ϕ	Yield function
ϕ_{ij}	Tensor defined in terms of δ_{ij} and ϵ_{ik}
ψ	Function of strain rate
η_{ij}	The Lagrangian strain function
ω_{jk}	Eulerian strain function

CHAPTER I

INTRODUCTION

Flights with manned spacecraft in the United States have in the past been terminated on water, although considerations have been given to recoveries on land. Furthermore, it is recognized that the craft would most probably impact on land rather than on water in the case of an abort immediately after lift-off from the launching pad. It is also known that flights to other planets will experience landings on solid or semi-solid materials. For these and other reasons it is of interest to study the characteristics of dynamic impact between solid bodies and soils.

It can be expected that there would exist a large range of angles of impact of the spacecraft with soils masses. For this reason it is of interest to determine impact characteristics for the full range of impact angles between horizontal and vertical. It is the purpose of this study to investigate the characteristics of horizontal impact although the procedures developed in this study could be used to analyze vertical as well as horizontal impact. With minor changes the procedures of the solution could be adapted for considerations of inclined impacts. These adaptations would involve the resolution of body forces from the weight of the soil into components parallel and normal to the direction of impact.

Research on dynamic loading on soils is evidenced by many published papers on the interaction between structural foundations and soils when the structure has been subjected to dynamic loading generated by an earthquake or by a blast. There is also available in the literature certain information on the effect of repeated loadings on soils. However, in the problem of horizontal impact on soils which is considered here, published analytical information is practically nonexistent.

In most soil mechanics problems dealing with stress distribution in a soil mass, the inertia effect in the soil has been neglected due to the fact that such problems deal with slowly applied loads, or the so-called "static" loads. In addition it is assumed that displacements within the mass are small and the material may be considered to behave elastically. With increasing rates of loading the static solution will continue to be valid only if displacements in the soil mass vary linearly with time, causing inertial forces to vanish. This would not be valid for nonlinear materials such as soils.

In some investigations the modulus of elasticity is allowed to vary with the strain level. These investigations consider the generalized Hooke's law, and the soil is considered to be nonlinearly elastic material.

The first objective of this study of load-deformation of soils is to consider the general case which takes into account inertial effects. The second objective is to consider the general case of finite deformation in contrast to the infinitesimal deformations. The third objective is to obtain a generalized modulus of deformation, E', and a generalized lateral strain ratio, ν', both depending on the strain level. A stress-strain analysis is developed to show that the nonlinear behavior of the material could be incorporated by E' and ν'. The fourth and final objective, which is a by-product of the first three, is to determine the response of the material to varying rates of loading.

This study considers plane strain only. All objects penetrating the soil are assumed to have infinite dimensions in a third direction, where strains are considered to be zero. The problem might be generalized to a three dimensional case but, for the method of solution used in this study, this generalization would require larger computer storage and excessive amounts of computer time.

A listing of the computer program is given together with a flow chart and input guide. Because of the large size of computer output, a complete output is not included. The significant results, however, are presented through several tables and figures.

CHAPTER II

STRESS-STRAIN RELATIONS

In the case of elastic, homogeneous, and isotropic bodies and with small deformations which satisfy the assumption of the linearized Hooke's Law, the system of stresses and strains are completely defined by two independent elastic constants. The modulus of elasticity E and Poisson's ratio ν are generally used as the constants, together with other conditions to insure uniqueness of the solution.

The purpose of this chapter is to consider the general case of stress-strain relationship, which will lead to the conclusion that for the system on hand, where finite deformations are considered, stresses and strains could also be defined by two functions E' and ν'. E' and ν' are functions of the state of strain. The term strain as used in this text refers to change in size and shape in general, whether recoverable or not. It will be also assumed that the system has an initial unstressed and unstrained state which implies the state of stress is a function of the state of strain only. The assumption of isotropy still holds. The system does not have to be homogeneous since the nature of the functions E' and ν' could be varied at different points inside the system using numerical techniques. Needless to say, the conditions of uniqueness of solution should also be satisfied.

In deriving a stress-strain relation, it can be written in general that stress is a function of strain,

$$\sigma_{ij} = f\ (\epsilon_{ij}) \tag{2-1}$$

where

σ_{ij} stands for the stress tensor and

ϵ_{ij} stands for the strain tensor.

Beginning with Eq 2-1, the reasoning suggested by Reiner (12) will be followed and it will be noted that Eq 2-1 involves the second rank stress tensor σ_{ij}. If it is attempted to develop the function $f\ (\epsilon_{ij})$, all the terms on the right hand side of Eq 2-1 must be mixed tensors of rank 2 multiplied by the inner products or scalers, and the general expansion of the function $f\ (\epsilon_{ij})$ would be,

$$\sigma_{ij} = F_0\ \delta_{ij} + F_1\ \epsilon_{ij} + F_2\ \epsilon_{kj}\ \epsilon_{ik} + F_3\ \epsilon_{kj}\ \epsilon_{sk}\ \epsilon_{is}$$

$$+ \ldots\ldots\ldots\ldots \text{ Infinite number of terms.} \tag{2-2}$$

F_0, F_1, F_2 are constants or functions of the strain invariants. The term δ_{ij} is known as Kronecker delta and it has the following values:

$\delta_{ij} = 0 \qquad\qquad i \neq j$

$\delta_{ij} = 1 \qquad\qquad i = j$

Making use of the Cayley-Hamilton Equation (9), it can be shown that:

$$\epsilon_{kj}\ \epsilon_{sk}\ \epsilon_{is} = \epsilon_{ij}\ J_3 - \epsilon_{ij}\ J_2 + \epsilon_{kj}\ \epsilon_{ik}\ J_1 \tag{2-3}$$

and therefore

$$\epsilon_{kj}\ \epsilon_{sk}\ \epsilon_{ms}\ \epsilon_{im} = \delta_{kj}\ \epsilon_{ik}\ J_3 - \epsilon_{kj}\ \epsilon_{ik}\ J_2 + \epsilon_{kj}\ \epsilon_{sk}\ \epsilon_{is}\ J_1$$

$$= \delta_{ij}\ J_1\ J_3 + \epsilon_{ij}\ (J_3 - J_1\ J_2) + \epsilon_{kj}\ \epsilon_{ik}\ (J_1^2 - J_2) \tag{2-4}$$

where

J_1, J_2, and J_3 are the first, second and third invariants of the strain tensor.

$$J_1 = \delta_{sk}\,\epsilon_{ks}$$

$$J_2 = -\,1/2\,\epsilon_{sk}\,\epsilon_{ks}$$

$$J_3 = 1/3\,\epsilon_{sk}\,\epsilon_{ms}\,\epsilon_{km}$$

Similarly the higher order terms can be expressed in terms of ϵ_{ij} ϵ_{ik} and ϵ_{kj} so that

$$\sigma_{ij} = F_0\,\delta_{ij} + F_1\,\epsilon_{ij} + F_2\,\epsilon_{kj}\,\epsilon_{ik} \tag{2-5}$$

F_0, F_1 and F_2 are different from those in Eq 2-2. F_0, F_1 and F_2 are either constants or functions of the three invariants of the strain tensor.

In this study consideration is given to active loading only, thus the unloading effect may be disregarded and Eq 2-5 can be written as:

$$\sigma_{ij} = F_1\,\epsilon_{ij} + F_2\,\epsilon_{kj}\,\epsilon_{ik} \tag{2-6}$$

Use is made of the relation

$$\epsilon_{kj}\,\epsilon_{ik} = \phi_{ij}\,J_3 - \delta_{ij}\,J_2 + \epsilon_{ij}\,J_1 \tag{2-7}$$

where the tensor ϕ_{ij} is defined as:

$$\phi_{kj}\,\epsilon_{ik} = \delta_{ij} \tag{2-8}$$

It can be shown for plane strain problems that J_3 vanishes, and Eq 2-6 becomes

$$\sigma_{ij} = F_1\,\epsilon_{ij} + F_2\,(\epsilon_{ij}\,J_1 - \delta_{ij}\,J_2)\ . \tag{2-9}$$

The stress σ_{ij} is resolved into its isotropic and deviatoric components;

$$\sigma = F_1{}' \frac{\epsilon_{ii}}{3} + F_2{}' \left(J_1 \frac{\epsilon_{ii}}{3} - \frac{J_2}{3} \right)$$

$$S_{ij} = K_1 \, (\epsilon'_{ij}) + K_2 \, (J_1 \, \epsilon'_{ij} - 2/3 \, J_2 \, \delta_{ij})$$

where

σ is the isotropic component of stress

S_{ij} is the deviatoric component of stress,

$$S_{ij} = \sigma_{ij} - \delta_{ij} \frac{\sigma_{ii}}{3}$$

ϵ'_{ij} is the deviatoric component of strain

$$\epsilon'_{ij} = \epsilon_{ij} - \delta_{ij} \frac{\epsilon_{ii}}{3}$$

then

$$\sigma_x = \sigma_{11} = (F_1{}' + F_2{}' \, J_1) \, (e) - F_2{}' \frac{J_2}{3}$$

$$+ K_1 \, (\epsilon_{11} - e) + K_2 \left[J_1 \, (\epsilon_{11} - e) - 2/3 \, J_2 \right]$$

$$\sigma_x = \left[(F_1{}' - K_1) + (F_2{}' - K_2) \, J_1 \right] e + (K_1 + K_2 \, J_1)$$

$$\epsilon_x - (F_2{}' + K_2) \, J_2 \qquad (2\text{-}10)$$

$$\sigma_y = \left[(F_1{}' - K_1) + (F_2{}' - K_2) \, J_1 \right] e + (K_1 + K_2 \, J_1)$$

$$\epsilon_y - (F_2{}' + K_2) \, J_2$$

$$\sigma_z = \left[(F_1{}' - K_1) + (F_2{}' - K_2) \, J_1 \right] e - (F_2{}' + K_2) \, J_2$$

$$\sigma_{xy} = (K_1 + K_2 J_1) \gamma_{xy} = \frac{(K_1 + K_2 J_1)}{2} \epsilon_{xy}$$

where

$$\gamma_{xy} = \frac{\epsilon_{xy}}{2}$$

$$\epsilon = \epsilon_x + \epsilon_y$$

$$F_1 \epsilon_{ij} = F_1' \frac{\epsilon_{ii}}{3} + K_1 \epsilon'_{ij}$$

$$F_2 \epsilon_{ij} = F_2' \frac{\epsilon_{ii}}{3} + K_2 \epsilon'_{ij}$$

$$F_2 = \frac{F_2'}{3} + (2/3) K_2$$

Simplifying the notations, Eq 2-10 is written as:

$$\sigma_x = \alpha e + 2 \beta \epsilon_x - \mathcal{K} J_2 \qquad \text{(a)}$$

$$\sigma_y = \alpha e + 2 \beta \epsilon_y - \mathcal{K} J_2 \qquad \text{(b)}$$

$$\sigma_z = \alpha e - \mathcal{K} J_2 \qquad \text{(c)} \qquad \text{(2-11)}$$

$$\sigma_{xy} = \beta \epsilon_{xy} \qquad \text{(d)}$$

where

$$\alpha = f(F_1', K_1', F_2', K_2', J_1) = f(J_1, J_2)$$

$$\beta = f(K_1, K_2, J_1) = f'(J_1, J_2)$$

$$\mathcal{K} = f(F_2', K_2) = f'''(J_1, J_2)$$

Equation 2-11 is similar to the classical equations of the theory of elasticity except that the constants are functions of the first and second invariant of the strain tensor. In comparison to the elasticity equations, α corresponds to Lame$'$ constant λ, and β corresponds to the shear modulus G.

Determination of E' and ν'

The understanding of deformation properties of soils is of great importance to this problem. Any rational solution, regardless of how sophisticated that solution might be, remains as a crude approximation if the deformation properties are not well known. One step taken in this study for understanding real soil behavior is the consideration of finite deformation. The next step was to obtain E' and ν' as a function of some deformation index. Since the two functions depend on the invariants of strain, then the ideal thing would be to obtain E' and ν' as functions of these invariants. Such process, however, is hard to achieve since it involves many parameters. The anatomy of finite deformation has been described by Reiner (12) and Novozhilov (10). Reiner obtained five parameters which are constants or functions of the invariants of strain tensor. Novozhilov obtained six physical constants. The relations described by Novozhilov take the following form for a plane strain case:

$$\sigma_x = \frac{E}{1+\nu}\Big\{ (1 + \gamma_1 J_1^2 + \gamma_2 J_2)\, \epsilon_x + \frac{\nu}{1-2\nu} J_1 + \gamma_3 J_1^3 - (2\gamma_1 + \gamma_2 + \gamma_4) J_1 J_2 + \gamma_4 J_1 \left(\epsilon_x^2 + 1/4\, \epsilon_{xy}^2 \right) \Big\} \qquad (2\text{-}12)$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\Big\{ (1 + \gamma_1 J_1^2 + \gamma_2 J_2)\, \epsilon_{xy} + \gamma_4 J_1 \Big[(\epsilon_x + \epsilon_y)\, \epsilon_{xy} \Big] \Big\} \qquad (2\text{-}13)$$

E, ν, γ_1, γ_2, γ_3, γ_4 are six physical constants of which the last five are dimensionless.

$$J_1 = \epsilon_x + \epsilon_y$$

$$J_2 = \epsilon_x \, \epsilon_y - 1/4 \, (\epsilon_{xy}{}^2)$$

It can be noticed that the definition of σ_x in Eq 2-13 is similar to that of Eq 2-11 if it is considered that

$$\left(\frac{E}{1+\nu} \right) (1 + \gamma_1 \, J_1{}^2 + \gamma_2 \, J_2) = 2 \, \beta \, (J_1, \, J_2) \tag{2-14}$$

and

$$\left(\frac{E}{1+\nu} \right) \left(\frac{\nu}{1-2\nu} \right) J_1 = \alpha \, (J_1, \, J_2) \tag{2-15}$$

and

$$\frac{E}{1+\nu} \left(\gamma_3 \, J_1{}^3 - (2 \, \gamma_1 + \gamma_2 + \gamma_4) \, J_1 \, J_2 + \gamma_4 \, J_1 \, (\epsilon_x{}^2 + 1/4 \, \epsilon_{xy}{}^2)^2 \right.$$

$$= - \mathcal{K} \, (J_1, \, J_2) \tag{2-16}$$

The constants E and ν multiplied by some function of J_1 and J_2 as in Eqs 2-14 and 2-15 are defined by new functions E' and ν'. The functions E' and ν' depend on the state of deformation and hence on the invariants J_1 and J_2. Therefore β and α are defined as:

$$\beta = \frac{E'}{2(1+\nu')} = G \tag{2-17}$$

$$\alpha = \frac{E' \;\; \nu'}{(1+\nu')(1-2\nu')} = \lambda \tag{2-18}$$

The term J_2 is a measure of octahedral shear strain. This quantity is assumed to be small enough so that $\mathcal{K} \; J_2$ in Eq 2-11 is eliminated. This assumption is justified for stresses below yielding.

Equation 2-11 can therefore be written as

$$\sigma_x = \lambda e + 2G \epsilon_x \quad \text{(a)}$$

$$\sigma_y = \lambda e + 2G \epsilon_y \quad \text{(b)}$$

$$\sigma_z = \lambda e \quad \text{(c)} \qquad (2\text{-}19)$$

$$\sigma_{xy} = G \epsilon_{xy} \quad \text{(d)}$$

where λ and G are defined as in Eqs 2-17 and 2-18.

Reiner's five parameters or Novozhilov's six parameters are difficult to obtain in the laboratory. One way to approximate reality is to obtain E' and ν' as functions of the axial strain ϵ_x. Many factors affect the deformation behavior of soil. Among those many factors are:

1. Type of soil
2. Moisture content
3. Confining pressure
4. Density
5. Rate of loading
6. Type of test
7. Grain size.

Experimental tests reported by Barkan (2) indicated that the modulus of deformation for sands does not depend on the moisture content or grain size. In this study the rate of loading may have some effect on the deformation functions E' and ν' since the time period during which the load is applied varies from low to high rates of loading. In this analysis the values used for the functions E' and ν' are those for low rate (static) loading corresponding to a strain rate of 0.625% per minute. The confining pressure and density also have a marked effect on E' and ν'. Ghazzaly and Dawson (3) obtained E' and ν' as functions of axial strain for sands of densities 94 pcf, 102 pcf and 108 pcf. For each of those densities four states of confining pressures were used.

Since the above curves were obtained for a low strain rate, the strain rate effect, or in other words, the inertia effect, can be neglected and the values are considered to correspond to the so-called static test. A truly static test corresponds to zero rate of loading which is practically impossible to obtain, and it can be concluded that the inertia of the specimen exists in any test. For a nonlinear material such as soil the rate of loading is expected to have a marked effect on the deformation curves which implies that for each rate of loading analyzed, a different deformation curve has to be used. Any dynamic test, that is for high rates of loading, should consider the inertia of the specimen. If the mass of the test apparatus is large, the inertia of the apparatus should also be considered.

A detailed discussion of the factors which affect the function E' and ν' are discussed in Ref. (3). The main purpose of this section is to point out the uncertainties involving these two functions. The determination of E' and ν' was based entirely on the work done by Ghazzaly and Dawson which is described in Ref. (3).

The ultimate load capacity as used in this analysis refers to the state of deformation at which material resistance decreases or ceases to increase with further deformation applied at the boundary. The process to determine the ultimate resistance is simply to obtain the resistance at each time until the load deformation curve becomes flat or changes the sign of slope. It is therefore assumed that the modulus of deformation E' and the lateral strain ratio ν' represent the material behavior at any state of deformation.

A sixth order polynomial was used to describe the modulus of deformation vs. axial strain curve and a fourth order polynomial was used to describe the lateral strain ratio vs. axial strain curve. Both curves correspond to a confining pressure of 2.32 psi. The range of strains used are from 0.001 to 0.1 in/in. for a sand with a density of 102 pcf. The curves are shown in Figs. 2-1 and 2-2. The curves indicate that by increase in deformation the modulus of deformation E' will decrease while the lateral strain ratio ν' will increase. A least-squares curve-fit program was used, and therefore, for a particular strain, the relations take the following forms:

$$E' = a_0 + a_1 \epsilon_x + a_2 \epsilon_x^2 + a_3 \epsilon_x^3 + a_4 \epsilon_x^4 + a_5 \epsilon_x^5 + a_6 \epsilon_x^6 \qquad (2\text{-}20)$$

$$\nu' = b_0 + b_1 \epsilon_x + b_2 \epsilon_x^2 + b_3 \epsilon_x^3 + b_4 \epsilon_x^4 \qquad (2\text{-}21)$$

where

E' is the modulus of deformation

ν' lateral strain ratio.

The coefficients of the above equations depend on the type of soil considered. For the material under study the following values were used:

$a_0 = 2.7139 \times 10^3$ $\qquad b_0 = 23.875 \times 10^{-2}$

$a_1 = -27.227 \times 10^4$ $\qquad b_1 = 46.742$

$a_2 = 13.213 \times 10^6$ $\qquad b_2 = -19.77 \times 10^2$

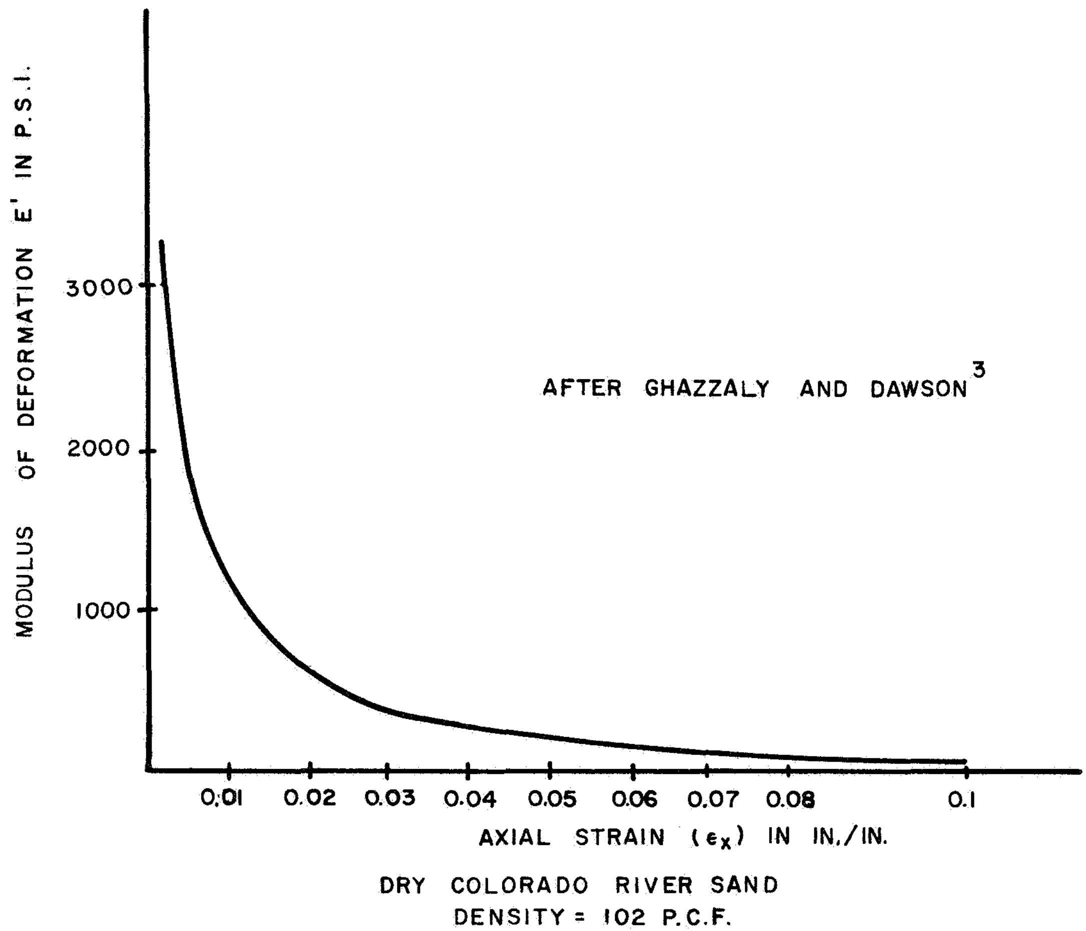


FIG. 2-1

MODULUS OF DEFORMATION

VS. AXIAL STRAIN

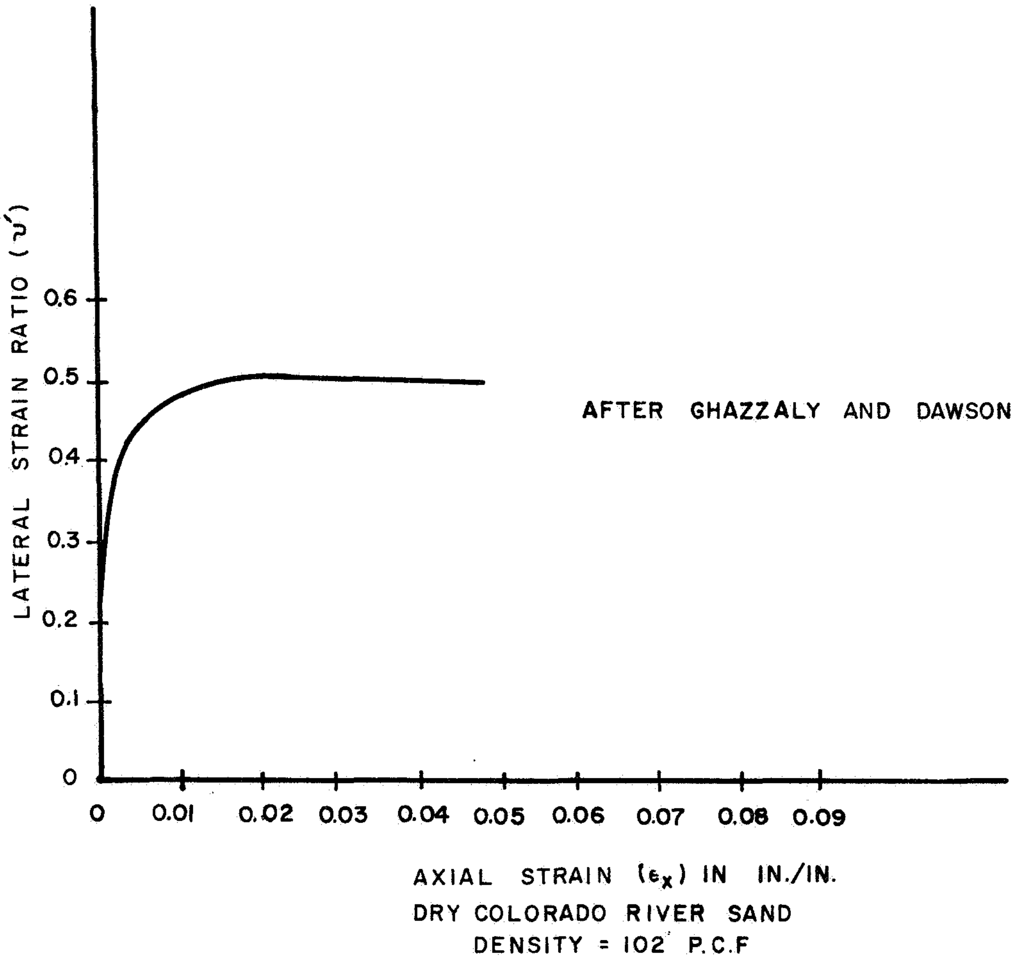


FIG. 2-2

LATERAL STRAIN RATIO VS. AXIAL STRAIN

$a_3 = -32.679 \times 10^7$ $\qquad$ $b_3 = 29.88 \times 10^3$

$a_4 = 41.633 \times 10^8$ $\qquad$ $b_4 = -14.841 \times 10^4$

$a_5 = -25.851 \times 10^9$

$a_6 = 61.38 \times 10^9$

The maximum value of E' could not be more than 2713.9 psi and the minimum value of ν' could not be less than 0.23875 or more than 0.5.

Strains*

The strains ε_x, ε_y, ε_{xy} are defined as follows:

$$\varepsilon_x = u_x - 1/2\,(u_x^2 + v_x^2) \qquad \text{(a)}$$

$$\varepsilon_y = v_y - 1/2\,(v_y^2 + u_y^2) \qquad \text{(b)} \qquad \text{(2-22)}$$

$$\varepsilon_{xy} = (u_y + v_x) - (u_y\, u_x + v_y\, v_x) \qquad \text{(c)}$$

where u and v are the displacements in the x and y directions. The above definitions arise from the actual state of deformation in a plane strain case.

Equilibrium

In the problem under consideration the displacements are specified at the boundary at any instant of time; therefore for a unique solution the conditions of equilibrium have to be satisfied, namely

$$\sigma_{xx,x} + \sigma_{xy,y} + F_x = \rho\, \ddot{u}$$

*Definitions are derived in Appendix A.

$$\sigma_{yy,y} + \sigma_{xy,x} + F_y = \rho \ddot{v}$$

or

$$\frac{\partial}{\partial_x} \left\{ (\lambda + 2G) \left[u_x - 1/2 \ (u_x^2 + v_x^2) \right] \right.$$

$$\left. + \lambda \left[v_y - 1/2 \left[(v_y^2 + u_y^2) \right] \right] \right\}$$

$$+ G \frac{\partial}{\partial_y} \left[(u_y + v_x) - u_y u_x - v_y v_x \right] = \rho \ddot{u} - F_x$$

and

$$\frac{\partial}{\partial_y} \left\{ (\lambda + 2G) \left[v_y - 1/2 \ (v_y^2 + u_y^2) \right] \right.$$

$$\left. + \lambda \left[u_x - 1/2 \ (u_x^2 + v_x^2) \right] \right\}$$

$$+ G \frac{\partial}{\partial_x} \left[(u_y + v_x) - u_y u_x - v_y v_x \right] = \rho \ddot{v} - F_y$$

Carrying out the differentiation of the above two equations, they can be written as,

$$(\lambda + 2G) \left[u_{xx} (1 - u_x) - v_x v_{xx} \right] + (\lambda + G) \left[v_{xy} (1 - v_y) \right.$$

$$\left. - u_y u_{xy} \right] + G \left[u_{yy} (1 - u_x) - v_{yy} v_x \right] = \rho \ddot{u} - F_x \qquad (2\text{-}23a)$$

$$(\lambda + 2G) \left[v_{yy} (1 - v_y) - u_y u_{yy} \right] + (\lambda + G) \left[u_{xy} (1 - u_x) \right.$$

$$\left. - v_x v_{xy} \right] + G \left[v_{xx} (1 - v_y) - u_{xx} u_y \right] = \rho \ddot{v} - F_y \qquad (2\text{-}23b)$$

Equations 2-23, a and b should be satisfied at each point inside the region, subjected to certain boundary conditions. In the case of infinitesimal deformation the strain products in Eq 2-23 would vanish. The resulting equations would be linear. The form of such equations is derived in Appendix B.

CHAPTER III

BOUNDARY CONDITIONS FOR TYPICAL PROBLEMS

It has been stated that the purpose of this study is to obtain the displacement and stress distribution in a soil mass due to the penetration of a rigid body at the contact surface. The body is assumed to be infinitely long in the Z direction. No displacements are allowed in the Z direction and all the problems to be solved are of the plane-strain type of problem. The penetrating body could be a plate, wedge or a cylinder.

Figures 3-1 through 3-6 illustrate six different problems. A plate is used to represent the rigid body in these figures. The plate is assumed very thin and infinitely rigid. In all these problems the contact surface is S. The bearing length L of the contact surface is constant for the case of the plate. The bearing length L changes at each stage of deformation in the case of a wedge or a cylinder as illustrated in Figs. 3-7 and 3-8. In all the problems, it is required to obtain the displacement and stress distribution throughout the region R. The sense of the coordinate system depends upon whether the rigid body is penetrating a vertical or a horizontal surface.

Problem 1. <u>Penetration of a rigid plate into a vertical surface, bounded by a horizontal surface at the edge of the plate, Fig. 3.1.</u>

The boundary conditions are:

Boundary Conditions	Locations	
$u = f(y, t)$	$x = 0$	$y \leq L$
$v = f(y, t)$	$x = 0$	$y \leq L$
$u = 0$ and $v = 0$	$x = 2L$	
$u = 0$ and $v = 0$		$y = 2L$

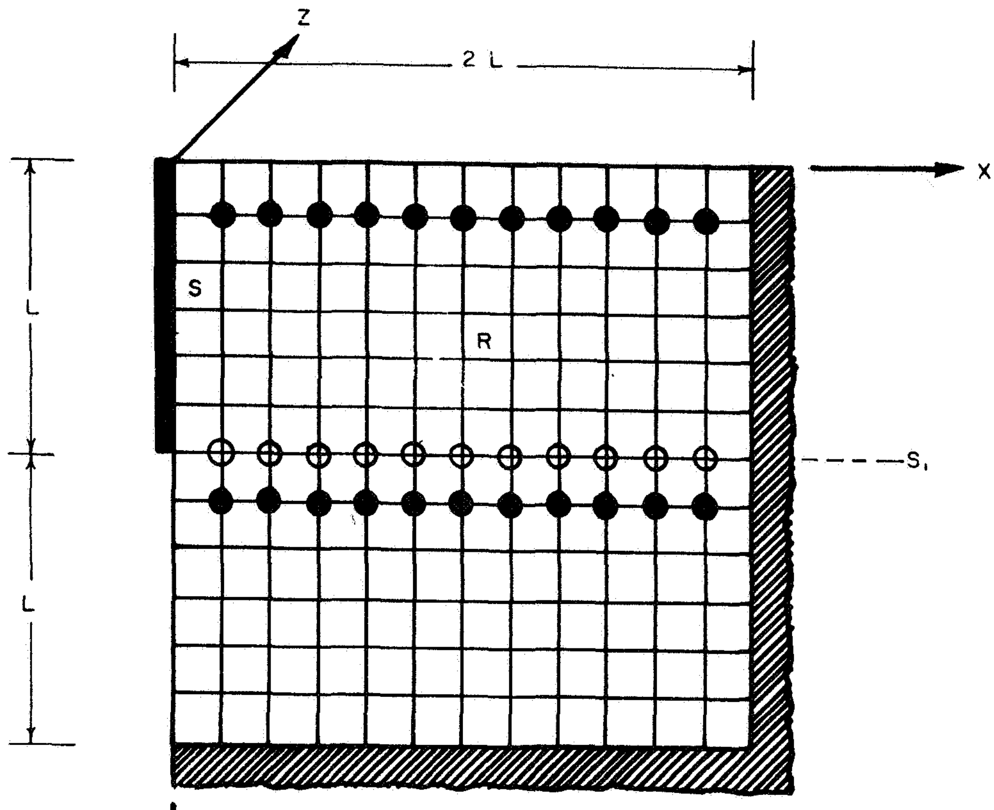


FIG. 3-1
PROBLEM 1

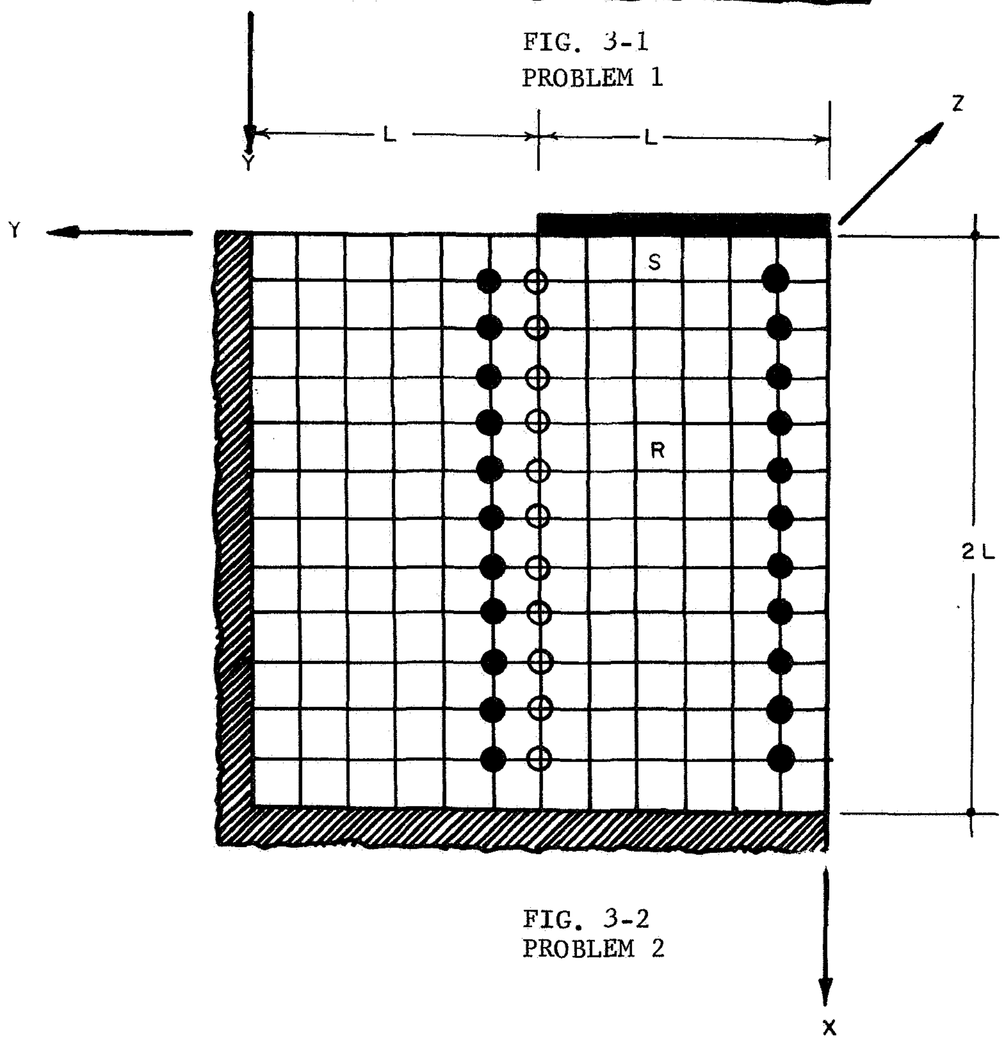


FIG. 3-2
PROBLEM 2

Boundary Conditions	Locations	
$\sigma_y = f(x, t)$		$y = 0$
$\sigma_x = 0$	$x = 0$	$y > L$

The length 2L could be increased and this gives more accurate results. Such increase would have little effect on the solution for points close to the contact surface S. The little increase in accuracy would be at the expense of computer time.

Problem 2. Penetration of a rigid plate into a horizontal surface, bounded by a vertical surface at the edge of the plate, Fig. 3-2.

The coordinates in problem 1 are turned 90 degrees clockwise, which means that the vertical surface in problem 1 becomes a horizontal surface in problem 2. In the actual solution the only difference is the direction of the body force. F_y (Eq 2-23b) in problem 1 is in the y direction, and stays as F_y in Eq 2-23b. In problem 2 the body force F_y becomes F_x and stays F_x in Eq 2-23a.

Boundary Conditions	Locations	
$u = f(y, t)$	$x = 0$	$y \leq L$
$u = 0$ and $v = 0$	$x = 2L$	
$u = 0$ and $v = 0$		$y = 2L$
$\sigma_y = f(x, t)$		$y = 0$
$\sigma_x = 0$	$x = 0$	

Problem 3. Penetration of a rigid plate into an infinitely long vertical surface, normal to the direction of loading. Displacements assumed to vanish at finite distances from both edges of the plates, Fig. 3-3.

The boundary conditions take the form:

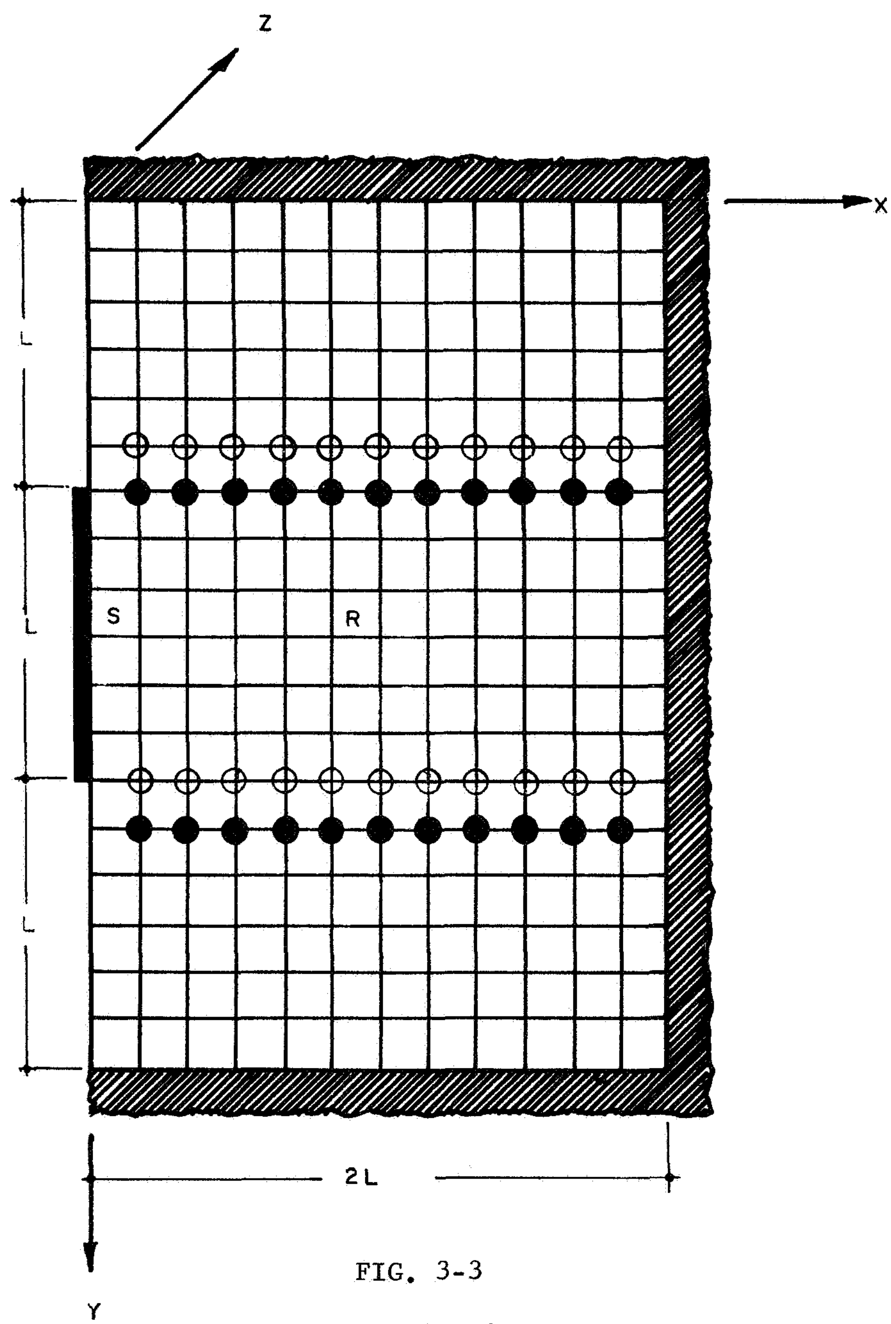


FIG. 3-3

PROBLEM 3

Boundary Conditions	Locations	
$u = 0$ and $v = 0$		$y = 0$
$u = 0$ and $v = 0$	$x = 2L$	
$u = 0$ and $v = 0$		$y = 3L$
$u = f\ (y,\ t)$ and $v = f\ (y,\ t)$	$x = 0$	$L \leq y \leq 2L$
$\sigma_x = 0$	$x = 0$	$L > y > 2L$

Problem 4. Penetration of a rigid plate into an infinitely long horizontal surface, normal to the direction of loading. Displacements assumed to vanish at finite distances from both edges of the plate, Fig. 3-4.

Problem 4 is the same as problem 3, except that the coordinates and the body force are turned 90 degrees clockwise.

Boundary Conditions	Locations	
$u = 0$ and $v = 0$		$y = 0$
$u = 0$ and $v = 0$	$x = 2L$	
$u = 0$ and $v = 0$		$y = 3L$
$u = f\ (y,\ t)$ and $v = f\ (y,\ t)$	$x = 0$	$L \leq y \leq 2L$
$\sigma_x = 0$	$x = 0$	$L > y > 2L$

Problem 5. Penetration of a rigid plate into a vertical surface, bounded by a horizontal surface at a distance L' from the edge of the plate, Fig. 3-5.

Boundary Conditions	Locations	
$u = f\ (y,\ t)$ and $v = f\ (y,\ t)$	$x = 0$	$L' < y \leq L + L'$
$u = 0$ and $v = 0$		$y = 2L + L'$
$u = 0$ and $v = 0$	$x = 2L$	
$\sigma_y = f\ (y,\ t)$		$y = 0$
$\sigma_x = 0$	$x = 0$	$y < L'$
$\sigma_x = 0$	$x = 0$	$y > L + L'$

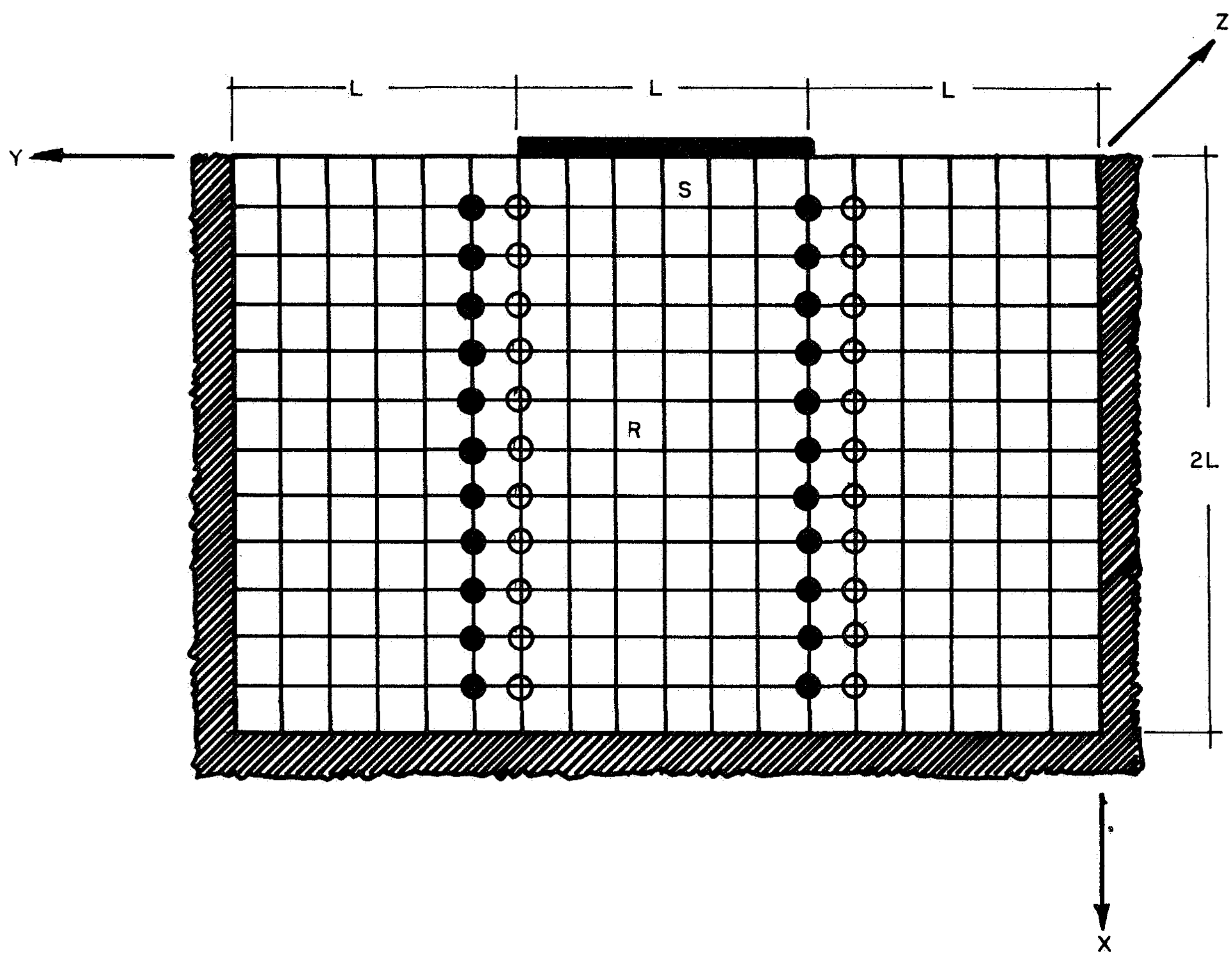


FIG. 3-4

PROBLEM 4

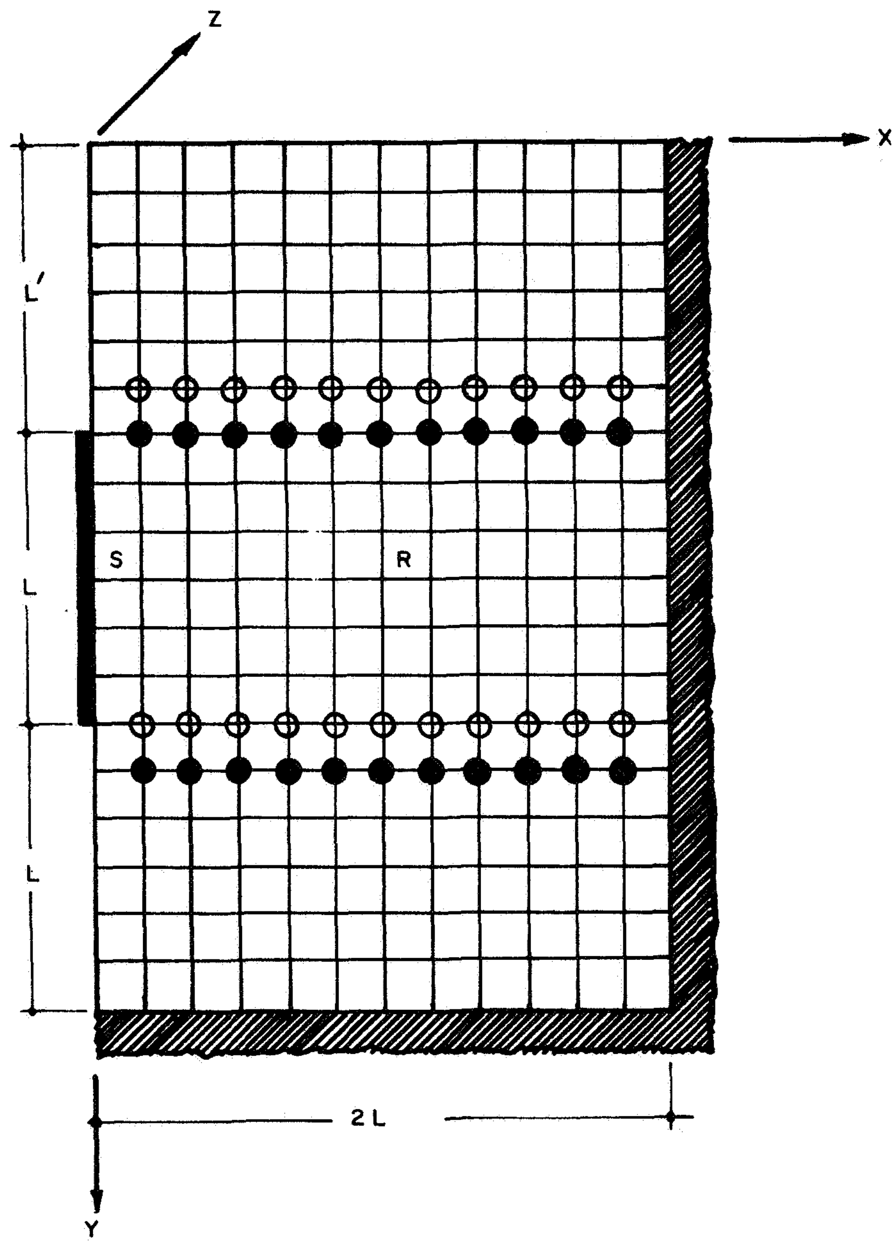


FIG. 3-5

PROBLEM 5

Problem 6. <u>Penetration of a rigid plate into a horizontal surface, bounded by a vertical surface at a distance L' from the edge of the plate, Fig. 3-6.</u>

Compared to problem 5, the coordinates and the direction of body force are turned 90 degrees clockwise.

<u>Boundary Conditions</u>	<u>Locations</u>	
$u = f\ (y,\ t)$ and $v = f\ (y,\ t)$	$x = 0$	$L' < y \leq L + L'$
$u = 0$ and $v = 0$		$y = 2L + L'$
$u = 0$ and $v = 0$	$x = 2L$	
$\sigma_x = 0$	$x = 0$	$y < L'$
$\sigma_x = 0$	$x = 0$	$y > L + L'$
$\sigma_y = f\ (x,\ t)$		$y = 0$

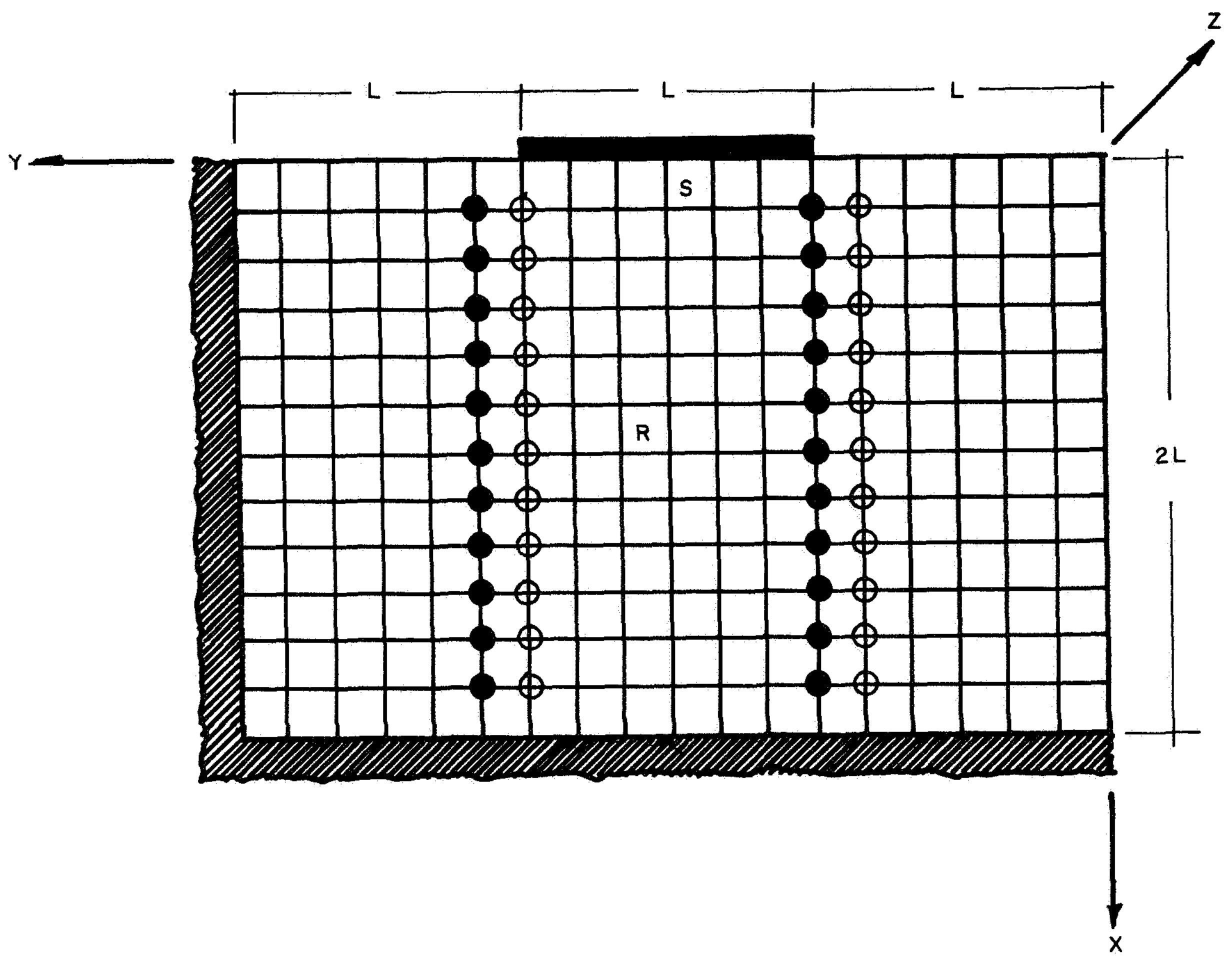


FIG. 3-6

PROBLEM 6

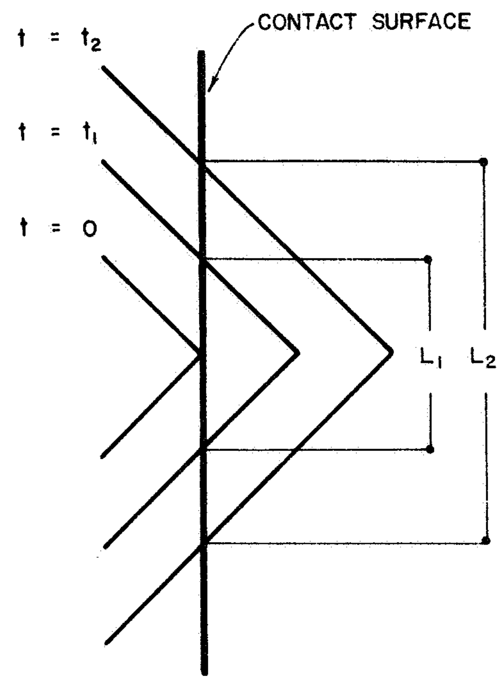


FIG. 3-7

PENETRATION
OF A WEDGE

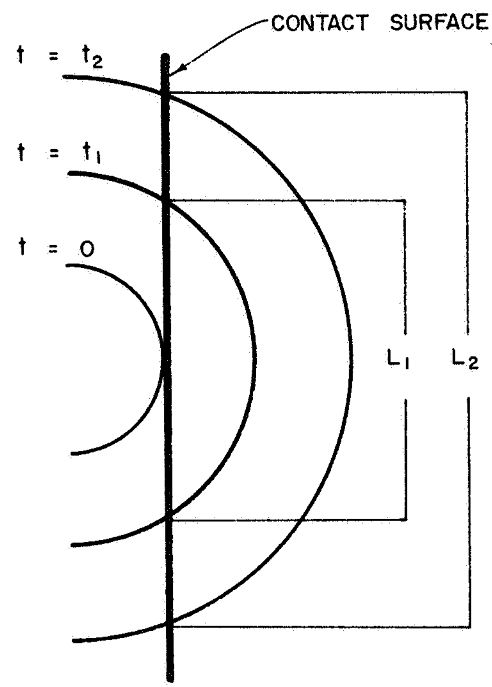


FIG. 3-8

PENETRATION
OF A CYLINDER

CHAPTER IV

METHOD OF SOLUTION

This chapter is concerned with the solution of problems outlined in Chapter III. The solution uses a numerical procedure based on finite difference approximations.

Equations 2-23 are the equilibrium equations written in terms of displacements; the horizontal displacement, u and the vertical displacement v. The method involves three basic steps.

1. The region R, indicated in Figs. 3-1 through 3-6, is covered with a mesh. Square or rectangular meshes can be used, the size of the mesh is prescribed by a horizontal increment length HX, and a vertical increment length HY. For square meshes HX and HY will be identical. The smaller the increment length HX and/or HY, the more accurate is the solution. The solution of differential equations in a finite difference form will approach the exact solution of the original differential equations when HX and HY approach zero.

Displacements on the boundary in contact with the penetrating object are specified at certain intervals of time. The intervals of time are specified by HT in seconds. In this analysis, displacements on the boundary are given as an increasing function of time, and at each instant of time, displacements in the region R are obtained. The system can be viewed as three dimensional in x, y and t coordinates where t stands for time, and the system starts with zero time. In the numerical solution the position of a nodal point 0 in the mesh is prescribed by the subscripts i, j and k, where i is the column number, j is the row number and k is the time number. Two fictitious time numbers; $k = 1$ and 2 are used to define the inertia force at the start of deformation, therefore for $k = 3$, the existing state of deformation corresponds to time HT. For $k = 4$ the deformation corresponds to the time 2 HT; similarly for $k = 10$, the time elapsed will be 8 HT, or in general, $t = (k-2)$ HT. The intersection of a particular row and column defines the material position of the nodal point 0. By material position, it is meant the X and Y coordinates of the nodal point 0. Row and column numbers start with 1 and therefore for a nodal point 0,

$$X = (i-1)\ HX \tag{4-1}$$

$$Y = (j-1)\ HY \tag{4-2}$$

The analysis can be used to study influence of a rigid inclusion in the region R. In this case, a specified number of adjacent nodal points should be prevented from moving. Such nodal points can be specified anywhere in the region R.

2. Equations 2-23a and b are written in a finite difference analogue. These equations are nonlinear difference equations. Figure 4-1 shows a typical nodal point together with the adjacent nodal points. If the material position of point 0 is i, j and the time position is k, then the position of the neighboring points are as shown in Fig. 4-1. Considering the convention as given in Fig. 4-1, the elements of Equations 2-23 can be written in finite difference form, as:

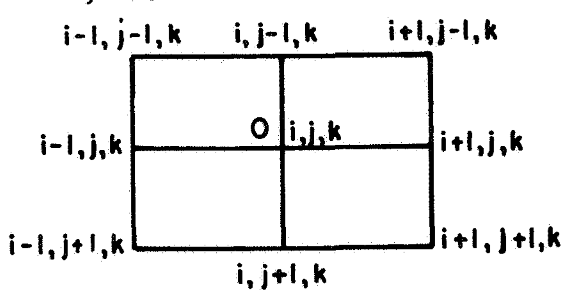


FIG. 4-1

FINITE DIFFERENCE CONVENTION

$$A = u_{xx} = (u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}) \ / \ HX^2$$

$$B = u_x = (u_{i+1,j,k} - u_{i-1,j,k}) \ / \ 2\ HX.$$

$$C = v_x = (v_{i+1,j,k} - v_{i-1,j,k}) \ / \ 2\ HX$$

$$D = v_{xx} = (v_{i-1,j,k} - 2v_{i,j,k} + v_{i+1,j,k}) \ / \ HX^2$$

$$F = v_y = (v_{i,j+1,k} - v_{i,j-1,k}) \ / \ 2\ HY$$

$$L = u_{xy} = (u_{i+1,j+1,k} - u_{i-1,j+1,k}) + u_{i-1,j-1,k} - u_{i+1,j-1,k}) \ / \ 4\ HX\ HY$$

$$M = u_{yy} = (u_{i,j-1,k} - 2u_{i,j,k} + u_{i,j+1,k}) \ / \ HY^2$$

$$N = v_{yy} = (v_{i,j-1,k} - 2v_{i,j,k} + v_{i,j+1,k})\ HY^2$$

$$PH = \ddot{u} = (u_{i,j,k-2} - 2u_{i,j,k-1} + u_{i,j,k}) \ / \ HT^2$$

$$PS = \ddot{v} = (v_{i,j,k-2} - 2v_{i,j,k-1} + v_{i,j,k}) \ / \ HT^2$$

$$Q = v_{xy} = (v_{i+1,j+1,k} - v_{i-1,j+1,k} + v_{i-1,j-1,k} - v_{i+1,\ j-1,k}) \ / \ 4\ HX\ HY$$

$$R = u_y = (u_{i,j+1,k} - u_{i,j-1,k}) \ / \ 2\ HY$$

$F_x = 0$ if the contact surface is vertical

$F_x = \gamma_s$ = unit weight (lb/cu. in.) if the contact surface is horizontal.

$F_y = \gamma_s$ = unit weight (lb/cu. in.) if the contact surface is vertical

$F_y = 0$ if the contact surface is horizontal

ρ = mass density (lb sec^2/in^4

PH and PS multiplied by ρ gives the inertia forces. PH and PS are written in backward difference form since the values of displacements at k+1 are not known.

The finite difference versions shown above are used throughout the region except where abrupt changes in displacement are anticipated or where there is no nodal point adjacent to the point considered. In such cases, simple differences are written in terms of points which are only one material increment length apart. Taking for example problem 1 in Chapter III, for all nodal points which are located at one vertical increment length below the line S_1, as shown on Fig.3-1, a forward difference form is used, then R, L and M take the forms

$$R = (u_{i,j+1,k} - u_{i,j,k}) \;/\; HY$$

$$M = (u_{i,j+2,k} - 2u_{i,j+1,k} + u_{i,j,k}) \;/\; HY^2$$

$$L = (u_{i+1,j+1,k} - u_{i+1,j+1,k} + u_{k-1,j,k}$$
$$- (u_{i+1,j,k}) \;/\; 2HX\ HY$$

For nodal points on S_1, a backward difference is used, then R, L and M take the form

$$R = (u_{i,j,k} - u_{i,j-1,k}) \;/\; HY$$

$$M = (u_{i,j-2,k} - 2u_{i,j-1,k} + u_{i,j,k}) \;/\; HY^2$$

$$L = (u_{i+1,j,k} - u_{i-1,j,k} + u_{i-1,j-1,k} - u_{i-1,j,k}) \;/\; 2\ HX\ HY$$

For the six problems considered in Figs. 3-1 through 3-6, nodal points with an open circle are those where backward differences is used, those with a dark circle are those where forward difference is used. Such modifications are only for R, L and M. For all other points central differences are used.

Having written the elements of Eqs 2-23 in finite difference form, then Eqs 2-23 can be written in the form:

$$(\lambda + 2G)\ (A\Big[\ (1 - B) - C\ D\ \Big] + (\lambda + G)\ \Big[\ Q\ (1 - F) - R\ L\ \Big] + G\Big[\ M\ (1 - B) - N\ C\ \Big] - PH\ (\rho) + F_x = REX_{i,j,k} \tag{4-1a}$$

$$(\lambda + 2G)\ \Big(\ N\ (1 - F) - R\ M\ \Big) + (\lambda + G)\ \Big(\ L\ (1 - B) - C\ Q\ \Big) + G\ \Big(\ D\ (1 - F) - R\ A\ \Big) - PS\ (\rho) + F_y = REY_{i,j,k} \tag{4-1b}$$

$REY_{i,j,k} = REX_{i,j,k} = 0$ if equilibrium is satisfied

$|REX_{i,j,k}| > 0$ if equilibrium is not satisfied

$|REY_{i,j,k}| > 0$ if equilibrium is not satisfied.

The nonlinear Eqs 4-1a and b have to be satisfied at each node. The deformation parameters are functions of the state of deformation and written in terms of the deformation modulus E'; and lateral strain ratio ν'.

3. Equations 4-1a and b have to be solved numerically. The point relaxation technique together with the two-variable technique are used in this study. Such techniques have been described by Allen (1), and applied to the solution of systems resulting in simultaneous linear equations. The mesh which covers

the region R is a relaxation net. At each node two variables have to be found, u and v; therefore two residuals are defined at each node; REX and REY. The object is to reduce these residuals. Two basic operations are performed at each node which involve the addition, separately, of unit increments. Each operation gives a relaxation pattern. One pattern shows the changes made to the residual REX and the other shows the changes which affect the value of REY. The relaxation patterns can be deduced from Eqs 4-1. At a typical node 0 we apply a unit increment, $\Delta u = 1$, to u. The same increment is applied to the neighboring nodes, and in the meantime it is assumed that no changes are affecting v. For the typical node 0, Fig. 4-1, a unit increment of $\Delta u = 1$ at the i,j,k, will produce a change in $REX_{i,j,k}$ equal to:

$$\Delta REX_{i,j,k} = (\lambda + 2G)\left(\frac{-2}{HX^2}(1 - 0) - 0\right] + (\lambda + G)\ (0)$$

$$+ G\left(\frac{-2\ (1 - 0)}{HY^2}\right] - \frac{\rho}{HT^2}$$

$$= -2\left(\frac{\lambda + 2G}{HX^2} + \frac{G}{HY^2}\right) - \frac{\rho}{HT^2} \qquad (4\text{-}2)$$

The change in REX at i,j,k due to increment $\Delta u = 1$ at i,j-1,k is equal to

$$(\lambda + 2G)\ (0) + (\lambda + G)\left(\frac{-1}{2HY}\ (0)\right) + G\left(\frac{1}{HY^2}\ (1 - 0)\right)$$

$$= \frac{G}{HY^2} \qquad (4\text{-}3)$$

The change in $REX_{i,j,k}$ caused by unit increment at i, j+1, k is equal to

$$(\lambda + 2G)\ (0) + (\lambda + G)\left(\frac{-1}{2HY}\ (0)\right) + G\left(\frac{1}{HY^2}\right) = \frac{G}{HY^2} \qquad (4\text{-}4)$$

The change in $REX_{i,j,k}$ due to $\Delta u = 1$ at $i+1,j,k$ is equal to

$$(\lambda + 2G) \left(\frac{1}{HX^2} \left(1 - \frac{1}{2HX}\right) \right)$$

$$+ (\lambda + G) \ (0) + G \ (0)$$

$$= (\lambda + 2G) \left(\frac{1}{HX^2} - \frac{1}{2HX^3} \right) \tag{4-5}$$

The change in residual produced at i,j,k due to $\Delta u = 1$ at $i-1,j,k$ is equal to

$$(\lambda + 2G) \left(\frac{1}{HX^2} + \frac{1}{2HX^3} \right) \tag{4-6}$$

denoting

$\lambda + 2G$	as	VDR
$\lambda + G$	as	VSR
G	as	SHM
ρ	as	RHO

then Eqs 4-2 through 4-6 give the relaxation pattern shown in Fig. 4-2.

Equations 4-2 through 4-6 are deduced from Eq 4-1a.

For the REY residuals another relaxation pattern is deduced from Eq 4-1b and shown in Fig. 4-3.

The aim of this relaxation procedure is to reduce the values of all residuals to zero, increments Δu and Δv are applied at each node in order to liquidate the corresponding residuals at i,j,k. The technique used in this study involves an iteration process which involves the following:

1. Choose initial values of u and v at each node in the relaxation net. Zero values are initiated in this analysis.

2. Compute the residuals REX and REY according to Eq 4-1a and b.

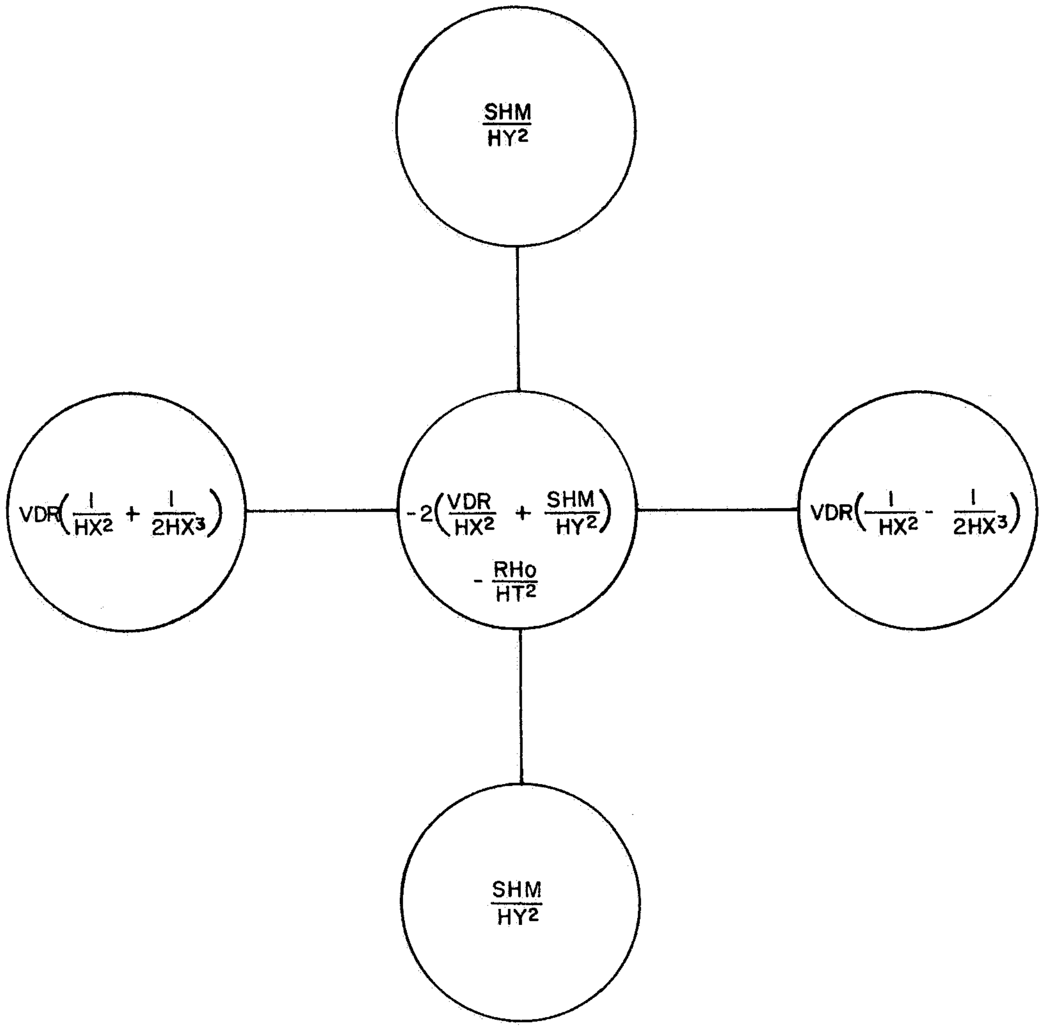


FIG. 4-2

RELAXATION PATTERN

LIQUIDATION OF REX RESIDUALS

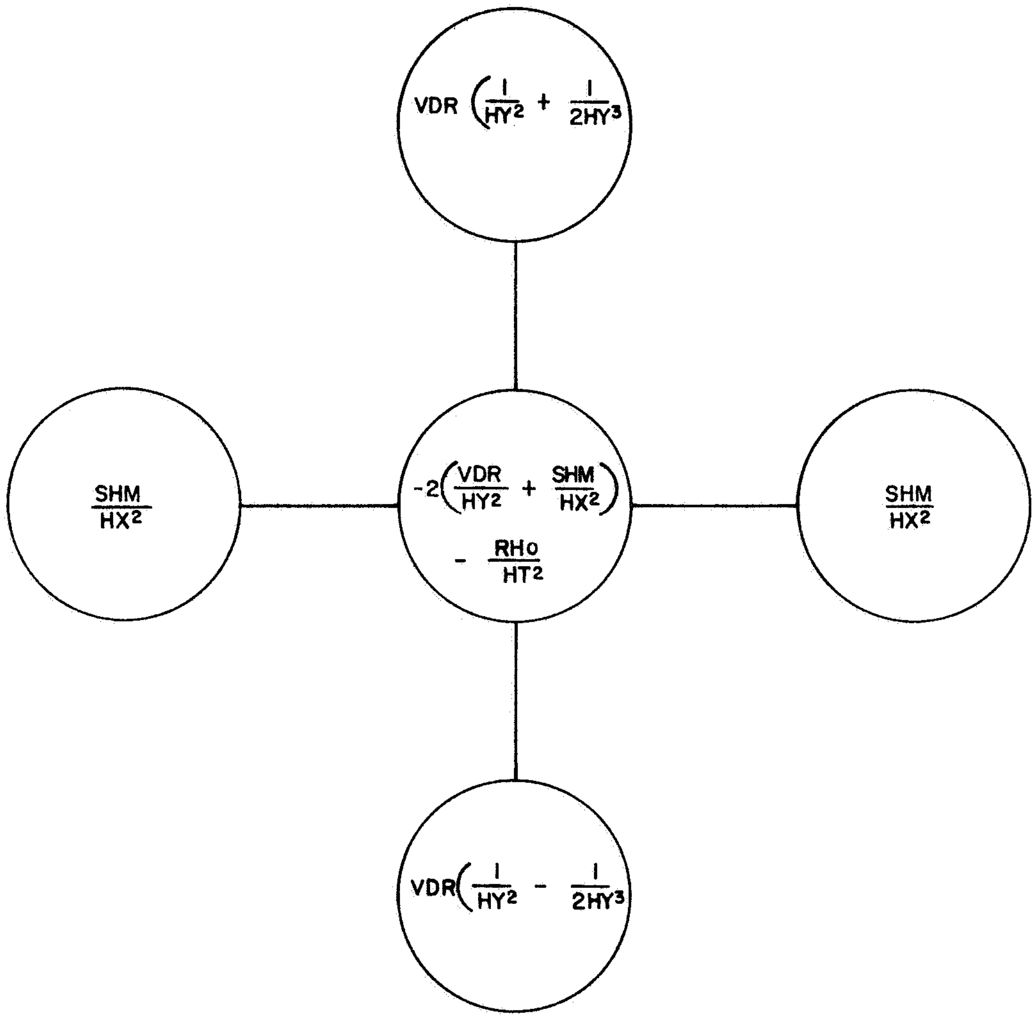


FIG. 4-3

RELAXATION PATTERN

LIQUIDATION OF REY RESIDUALS

3. Begin to liquidate the REX residuals. Using the relaxation pattern shown in Fig. 4-2, this is done by the application of increments Δu. At a particular node, Δu will be equal to $- \text{REX} / \left[-2 \left(\frac{\text{VDR}}{\text{HX}^2} + \frac{\text{SHM}}{\text{HY}^2} \right) - \frac{\text{RHO}}{\text{HT}^2} \right].$ Complete liquidation is not necessary at this point. A reduction down to 10 percent of the original values is sufficient. No residuals are carried to the boundary nodes.

4. Calculate the residuals REX and REY again, using the new values of u at the end of Stage 3.

5. Liquidate the residuals REY partially, using the relaxation pattern shown in Fig. 4-3. At a particular node, Δv will be equal to

$$- \text{REY} / \left[-2 \left(\frac{\text{VDR}}{\text{HY}^2} + \frac{\text{SHM}}{\text{HX}^2} \right) - \frac{\text{RHO}}{\text{HT}^2} \right].$$

6. Recalculate the residuals REX and REY using the values of v at the end of Stage 5, which incorporates all the changes made.

7. Continue liquidation of REX residuals as described in Stage 3.

8. Recalculate the REY residuals and REX residuals.

9. Continue liquidation of REY residuals as described in Stage 5.

Stages 6 to 9 are repeated until all residuals are reduced to a certain specified tolerance. Hence, the smaller the tolerance the larger is the time required to obtain a solution.

It is important that the relaxation patterns be used with consistency. That is, the same relaxation pattern should be used at each node. In Appendix B, the procedure is shown to be convergent.

Equation 4-1 is valid at each node inside the region R. For nodes which are located at one increment length from those boundaries where stresses are specified, Eq 4-1 is modified. The terms $\sigma_{xx,x}$ and $\sigma_{yy,y}$ in Eq 2-23 are written in different form to include the state of stress at the boundary; this is done as follows:

$$\sigma_{xx,x} = \frac{\sigma_{x_{i,j,k}} - \sigma_{x_{i-1,j,k}}}{HX}$$

$$= \frac{1}{HX}\left[(\lambda + 2G) \left(B - 1/2\ (B^2 + C^2) \right) + \lambda \left(F - 1/2\ (F^2 + R^2) \right) - \sigma_{x_{i-1,j,k}} \right]$$

σ_x at the boundary for all problems is zero and therefore:

$$\sigma_{xx,x} = \frac{1}{HX}\left[VDR\ (B - 1/2\ (B^2 + C^2) + \lambda \left(F - 1/2\ (F^2 + R^2) \right) \right].$$

and similarly;

$$\sigma_{yy,y} = \frac{1}{HY}\left[VDR \left(F - 1/2\ (F^2 + R^2) \right) + \lambda \left(B - 1/2\ (B^2 + C^2) \right) - \sigma_{y_{i,j-1,k}} \right].$$

Therefore the equations of equilibrium at the nodal points at one increment length from the surfaces where stress is specified are:

a. For nodes adjacent to where σ_y is specified:

$$(\lambda + 2G) \left(A\ (1 - B) - C\ D \right) + (\lambda + G) \left(Q\ (1 - F) - R\ L \right) + G \left(M\ (1 - B) - N\ C \right) - PH\ (\rho) + F_x = REX_{i,j,k} \quad (4\text{-}7a)$$

$$(1/HY \Big[(\lambda + 2G)\ (F - 1/2\ (F^2 + R^2)\ ) + \lambda\ (B - 1/2$$

$$(B^2 + C^2)\ ) - \sigma_{y_{i,j,k}} \Big] + G \Big[L\ (1 - B)$$

$$- A\ R + D\ (1-F) - Q\ C \Big] - \rho\ (PS) + F_y = REY_{i,j,k} \qquad (4\text{-}7b)$$

b. For nodes adjacent to where σ_x is specified as zero,

$$(1/HX) \Big[(\lambda + 2G)\ (B - 1/2\ (B^2 + C^2)\ ) + \lambda\ (F - 1/2$$

$$(F^2 + R^2) \Big] + G \Big[M\ (1 - B) - L\ G + Q\ (1 - F)$$

$$- N\ C \Big] - \rho\ (PH) + F_x = REX_{i,j,k} \qquad (4\text{-}8a)$$

$$(\lambda + 2G) \Big[N\ (1 - F) - R\ M \Big] + (\lambda + G) \Big[L\ (1 - B) - C\ Q \Big]$$

$$+ G \Big[D\ (1 - F) - R\ A \Big] - PS\ (\rho) + F_y = REY_{i,j,k} \qquad (4\text{-}8b)$$

In the previous discussion, Eq 2-19 was employed as relating stress and strain at a point inside the media. The relations expressed in Eq 2-19 were assumed to be valid at any level of strain. This, however, is not strictly true unless the material is below yielding (the so-called plastic case). For the analysis of the stress strain relations after yield, and for a proposed elasto-plastic analysis, the reader is referred to Appendix C.

CHAPTER V

RESULTS AND DISCUSSION

Geometry of the Problem

The geometry of the problem which has been selected for detailed studies is similar to that of Problem 1, Chapter III. It is the problem of a rigid plate penetrating a vertical surface bounded by a horizontal surface which is free of stresses. Figure 5-1 shows the relaxation net considered with the following

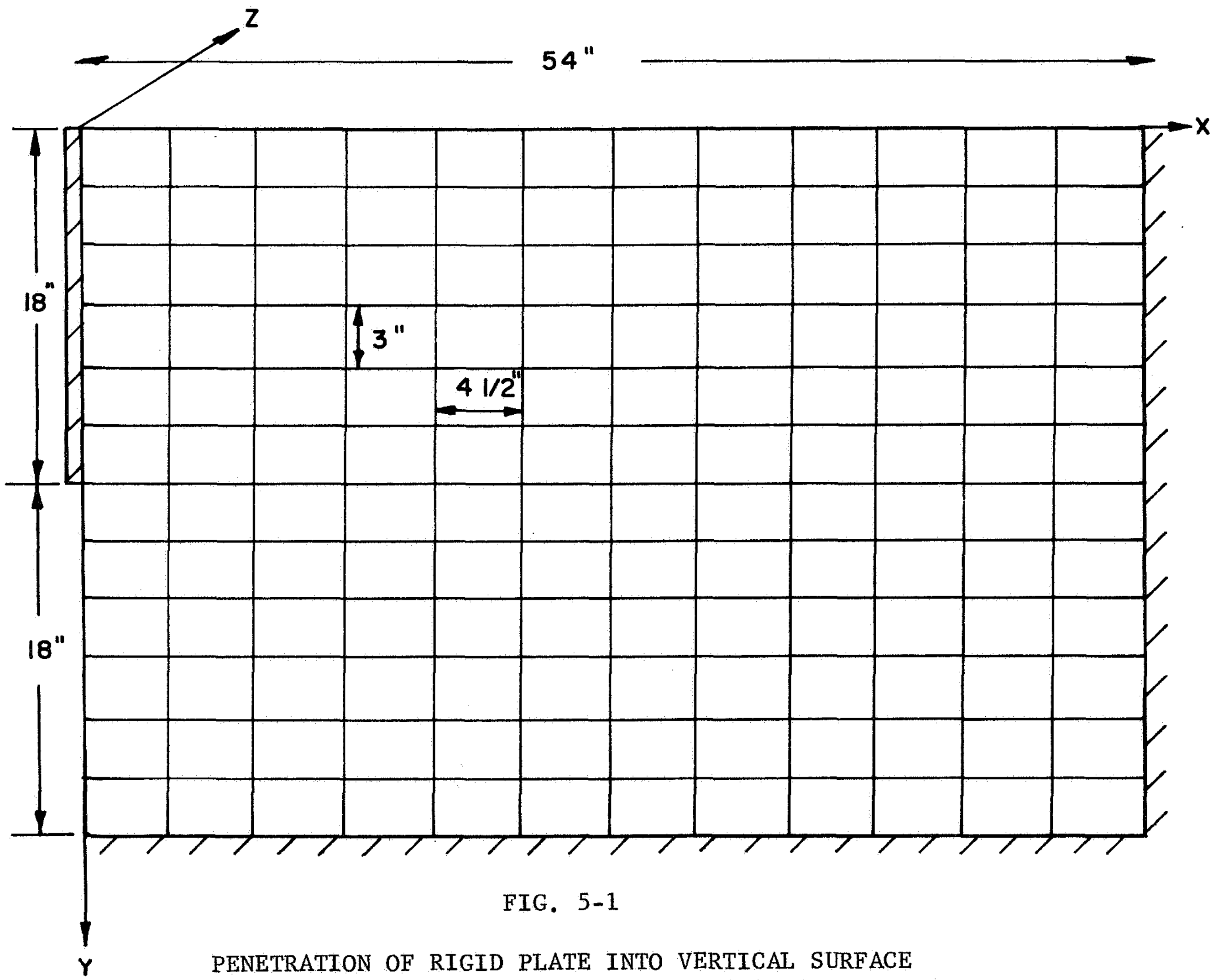


FIG. 5-1

PENETRATION OF RIGID PLATE INTO VERTICAL SURFACE BOUNDED BY A HORIZONTAL SURFACE AT THE EDGE OF PLATE

Boundary Conditions	Location	
$u = f\ (y,\ t)$	$x = 0$	$y \leq 18''$
$v = 0$	$x = 0$	$y \leq 18''$
$u = 0$ and $v = 0$	$x = 54''$	
$u = 0$ and $v = 0$		$y = 36''$
$\sigma_x = 0$	$x = 0$	$y > 18''$
$\sigma_y = 0$		$y = 0$

Width of plate = 12"

The function $f\ (y,\ t)$ is defined by the rate of loading. The movement of the plate is a rigid movement, that is, all nodes which lie on the boundary $x = 0$, $y \leq 18''$ move the same amount at a particular instant of time. For this particular problem, the function $f\ (y,\ t)$ is actually a function of time only or $f\ (t)$ and

$$u_o = RT \tag{5-1}$$

where

u_o is the movement of the plate

R is the rate of loading

T time elapsed since the start of penetration at $T = 0$.

Equation 5-1 shows that the movement of the plate varies linearly with time, and the velocity is equal to rate of loading. For a particular rate of loading the purpose was to determine the distribution of displacements and stresses throughout the medium included within the relaxation net, at specified time and displacement stations. Two methods can be used; either to fix the time stations and accordingly vary the value of displacement for different rates of loading or to fix the displacement stations and vary the value of the time interval for different rates of loading. Figure 5-2 shows the time displacement curves, for two rates of loading, 53.2 in./sec and 106.4 in./sec.

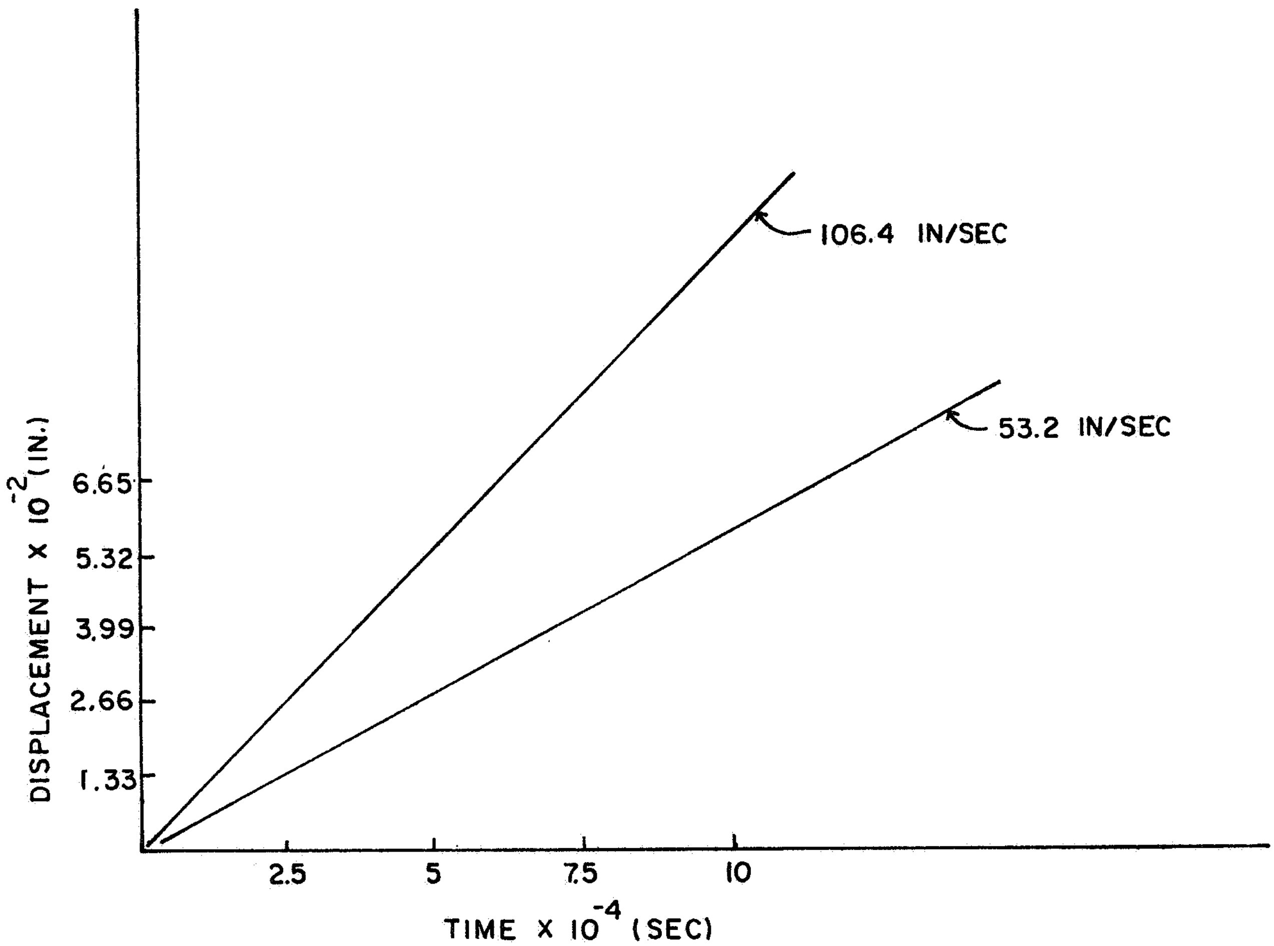


FIG. 5-2

EXAMPLES OF DISPLACEMENT-TIME CURVES AT THE BOUNDARY

The curves shown in Fig. 5-2 do not have to be linear. Any type of loading could be used as long as the displacement at each time station is known.

In solving for the two rates of loading the solution is carried out for the displacements of 0.0133, 0.260, and 0.0399 in., etc. The time interval between two consecutive displacements is 1.25×10^{-4} sec for a rate of loading of 106.4 in./sec, and 2.5 x 10^{-4} for the rate of loading of 53.2 in./sec. That is, for higher rates of loading, it takes less time to obtain a particular displacement.

Soil Properties

Soil properties were described in Ref. (3). The soil was a clean, light brown, dry sand known locally as Colorado River Sand. The sand was found to be subangular in shape and having a rather tough texture. Sand grains were mainly quartz, with some fragments of igneous, metamorphic and sedimentary rocks. The results of mechanical analysis reported in Ref. (3) are shown on a semi-logarithmic plot in Fig. 5-3. The sand was reported to have a specific gravity of 2.67 and maximum density of 108.26 pcf and a minimum density of 94 pcf. A curve for Ottowa Sand is shown for comparison.

The sand compacted to a density of 102 pcf which is a medium density condition, is considered in this study. The material is considered to have the deformation properties shown in Figs. 2-1 and 2-2.

Experimental Work

Experimental work related to this study has been reported by Horadam (7) and Hustad (8). The geometry of the problem related to the experimental work is different from that shown in Fig. 5-1. The 18 in. plate in the experimental work was treated as a short retaining wall, and hence the region $x = 0$, $y > 18''$ is not free from stresses. In the theoretical solution, the vertical displacements of all nodes in contact with the plate is assumed to be zero, which is not the true situation. The program, however, could be run by assuming that for a sufficiently fine relaxation net such displacements are fractions of the displacement of the neighboring nodes situated at one horizontal

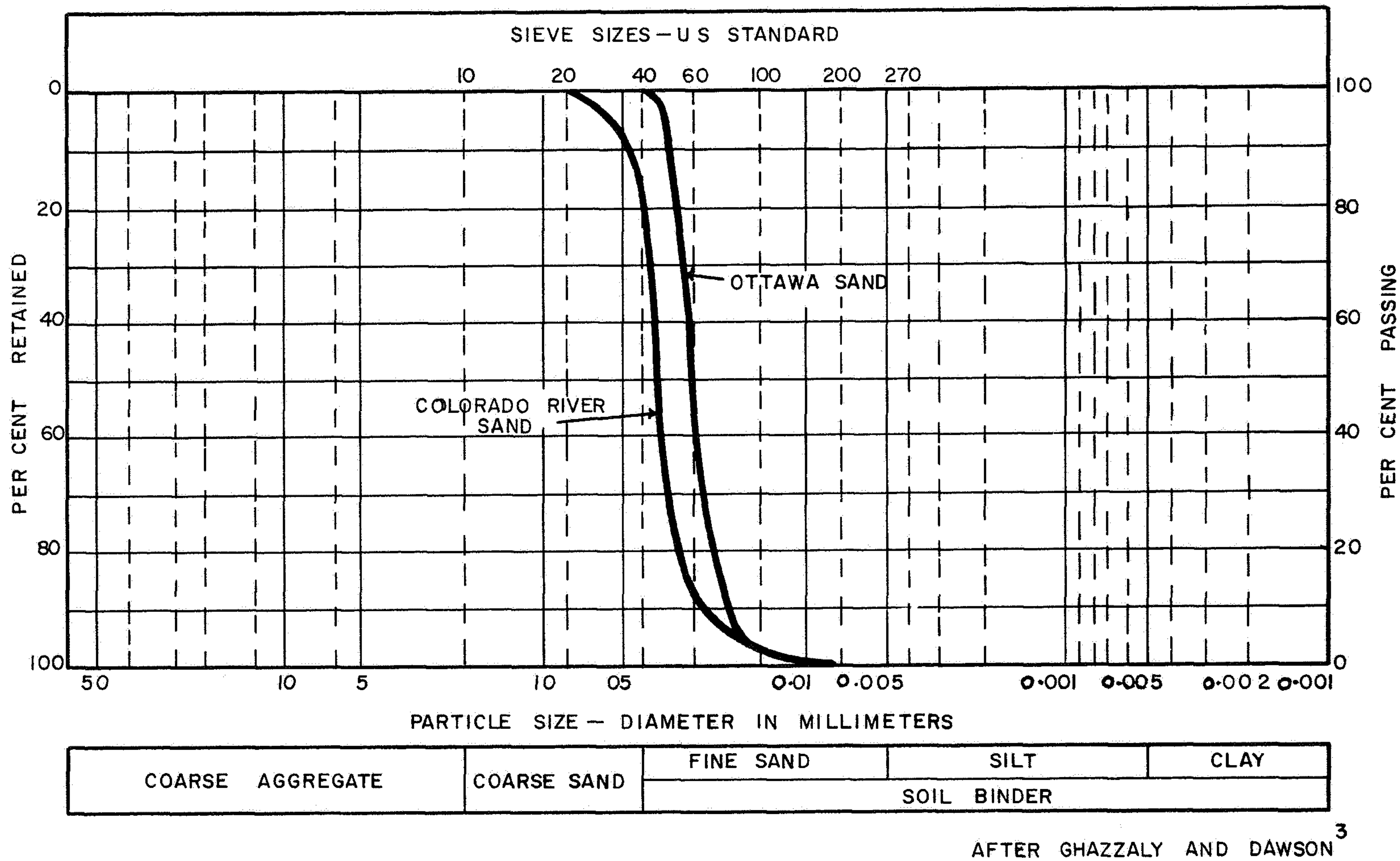


FIG. 5-3

MECHANICAL ANALYSIS

GRAIN SIZE ACCUMULATION CURVES

increment length. Such fractions could range from 0.0 to 1.0. The value of 1.0 represents a perfectly smooth plate.

In Ref. (7) slow rates of loading ranging from 0.0067 in./sec up to 2.66 in./sec were used. Figures 5-4 through 5-9 show the experimental load-displacement curves taken from Ref. (7) together with the theoretical curves developed in this study. The theoretical values are higher and the difference between theoretical and experimental increases with an increase in displacements. The values reported in the experimental curves are average values of several tests. In Ref. (7) maximum values of 1938 lbs were reported as well as values down to 1342 for the same rate of loading. This is due to the difficulty in repeating the same soil properties each time the test is run.

In Ref. (8) it was reported that "the wooden vertical restraint that was used to constrain the loading apparatus from moving in a vertical direction did not function properly", and hence the loading in the experimental work was not strictly horizontal. Some difficulties were encountered in trying to obtain a plane strain case in the experimental setup. A thorough discussion on such difficulties is treated in Ref. (8).

All the above mentioned factors, assumptions made in theoretical solutions, and the difference in the geometry would definitely contribute to difference between the theoretical and experimental values. Figure 5-10 shows the theoretical curves for six rates of loading up to 2.66 in./sec. Figure 5-10 illustrates that for slow rates of loading, the rate of loading has only a slight effect on the shape of the curves as well as on the ultimate resistance.

It was felt that at least one theoretical solution should have a common base with respect to the boundary conditions used in the experimental work reported in Refs. (7) and (8). For that purpose a theoretical solution

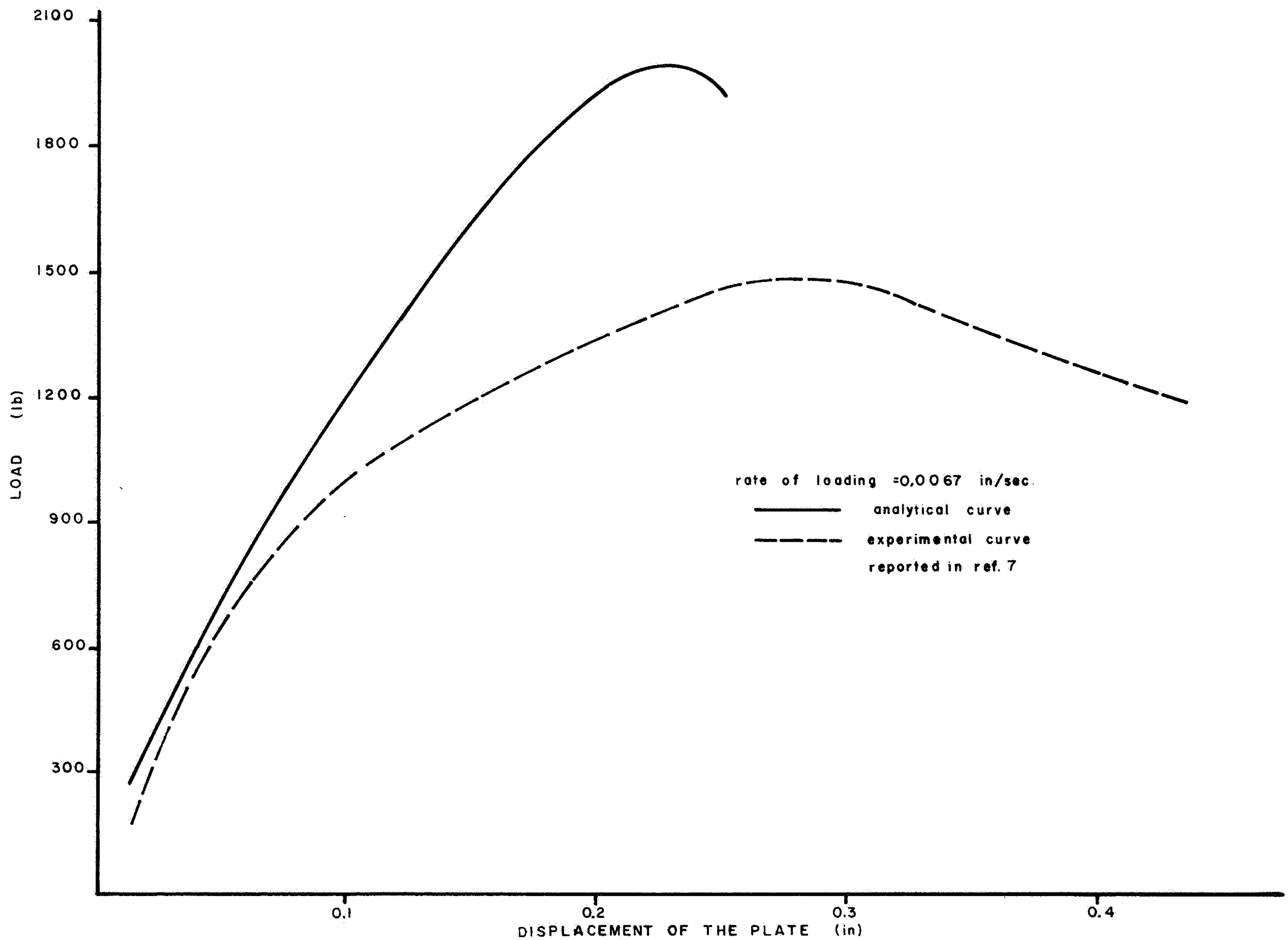


FIG. 5-4

LOAD VS. DISPLACEMENTS

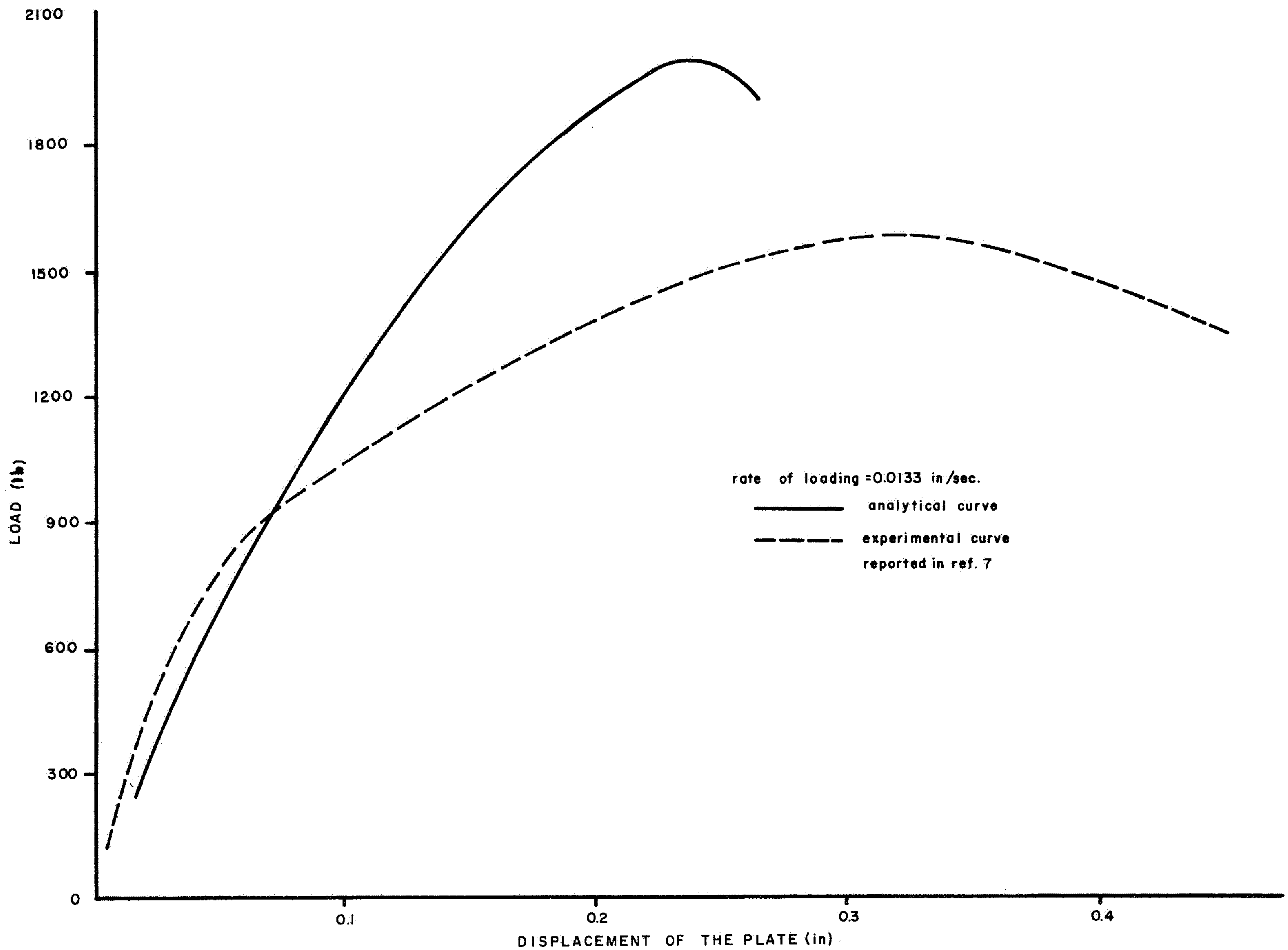


FIG. 5-5

LOAD VS. DISPLACEMENTS

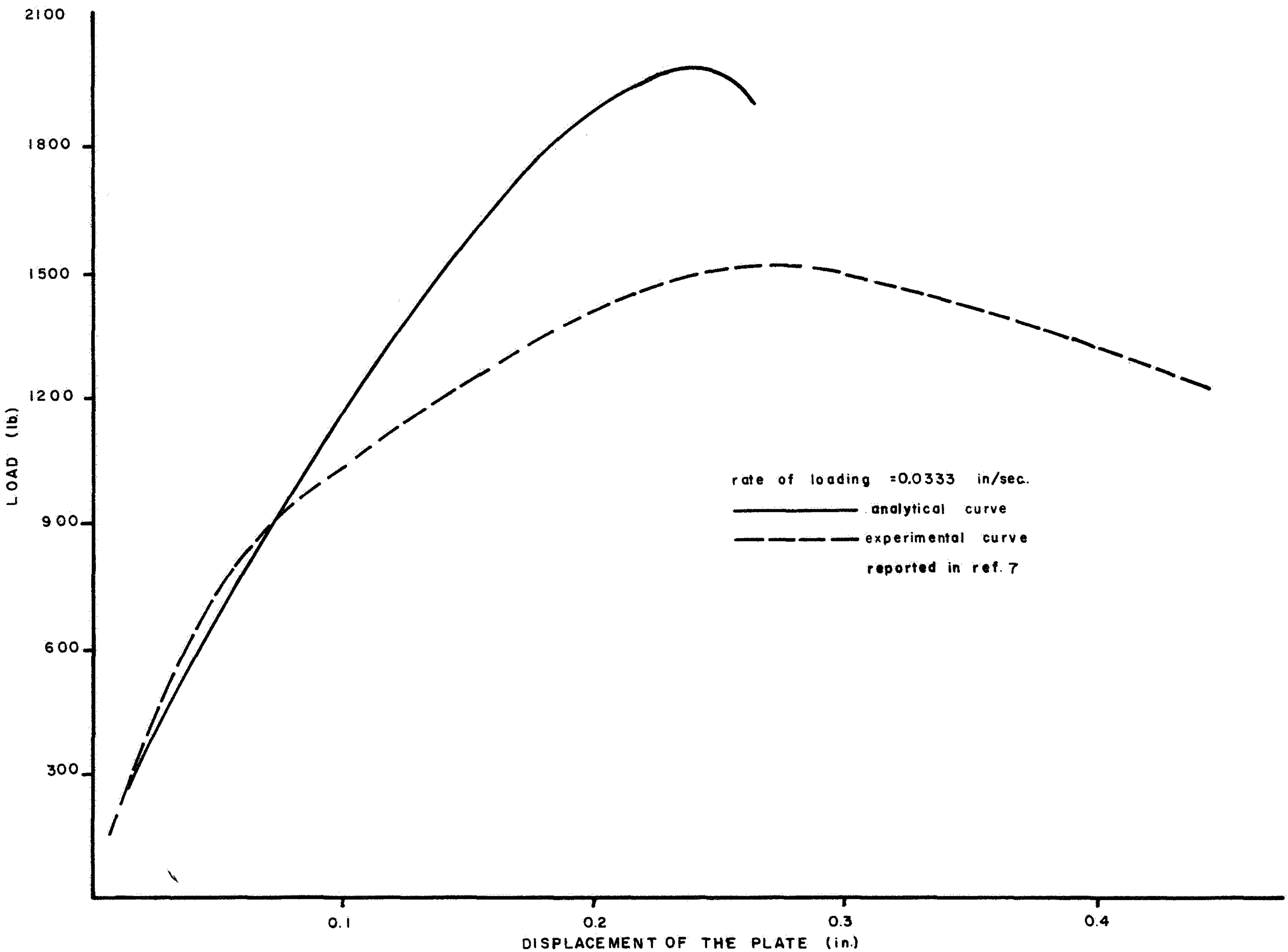


FIG. 5-6
LOAD VS. DISPLACEMENTS

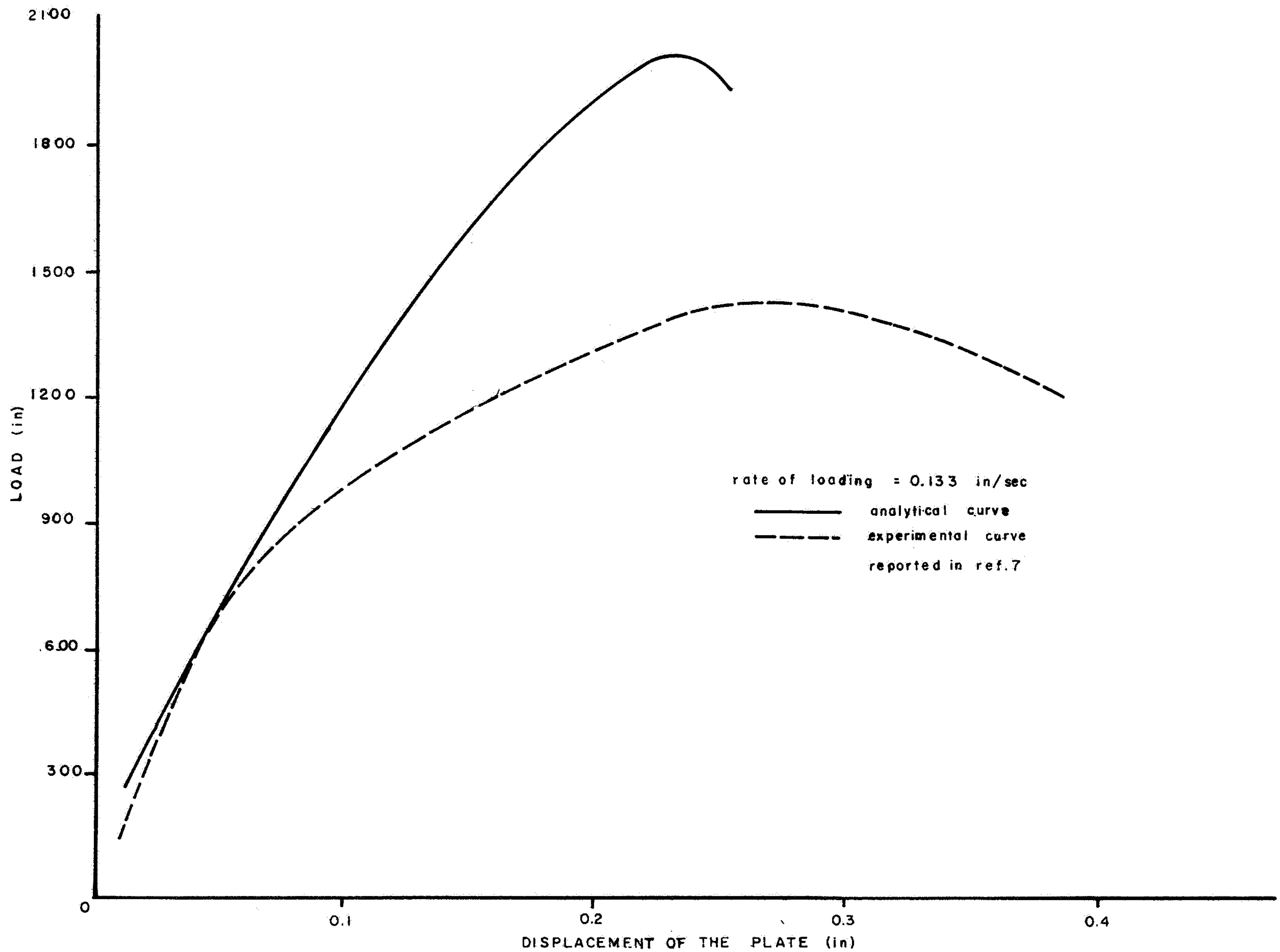


FIG. 5-7

LOAD VS. DISPLACEMENTS

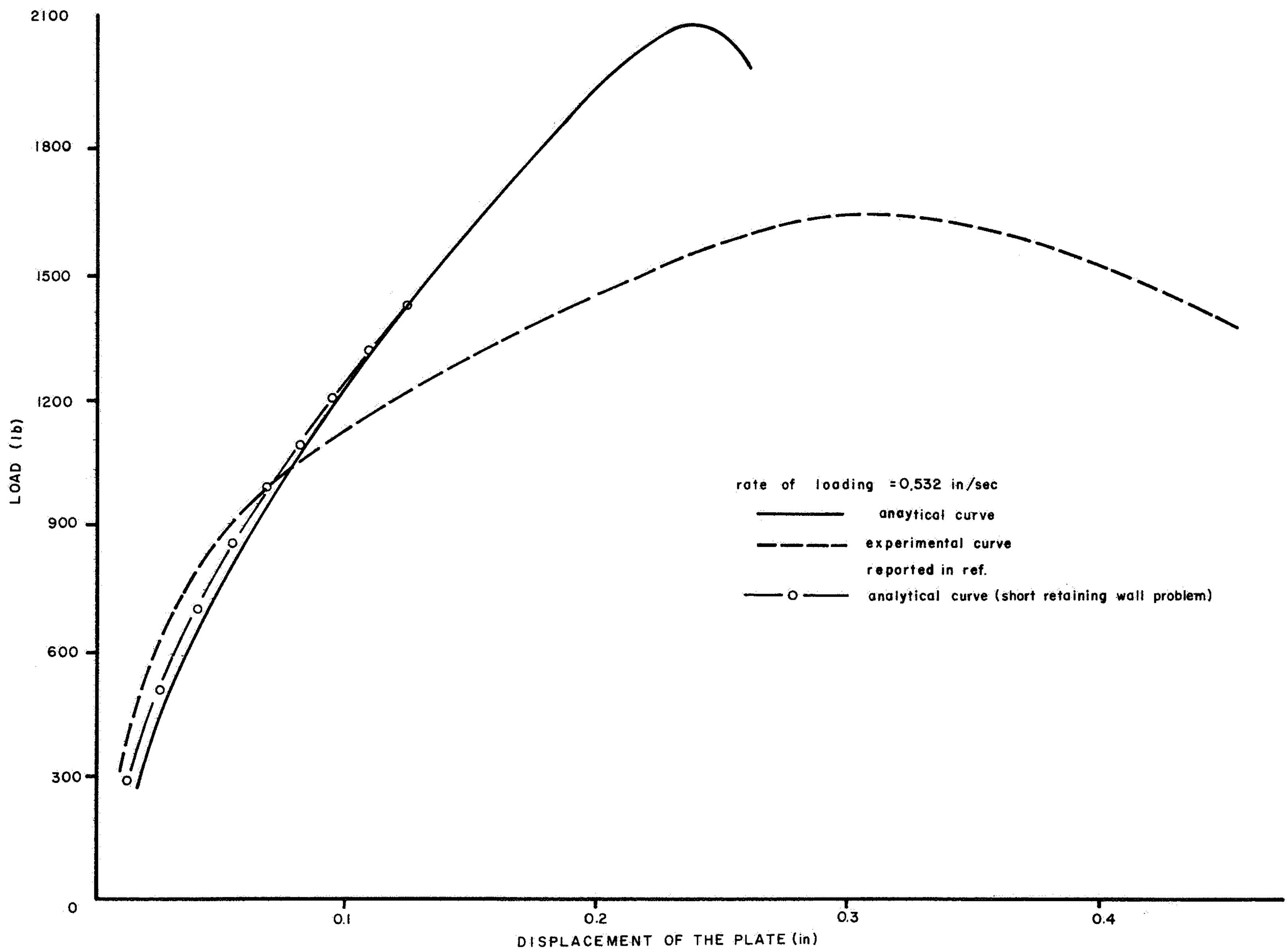


FIG. 5-8
LOAD VS. DISPLACEMENTS

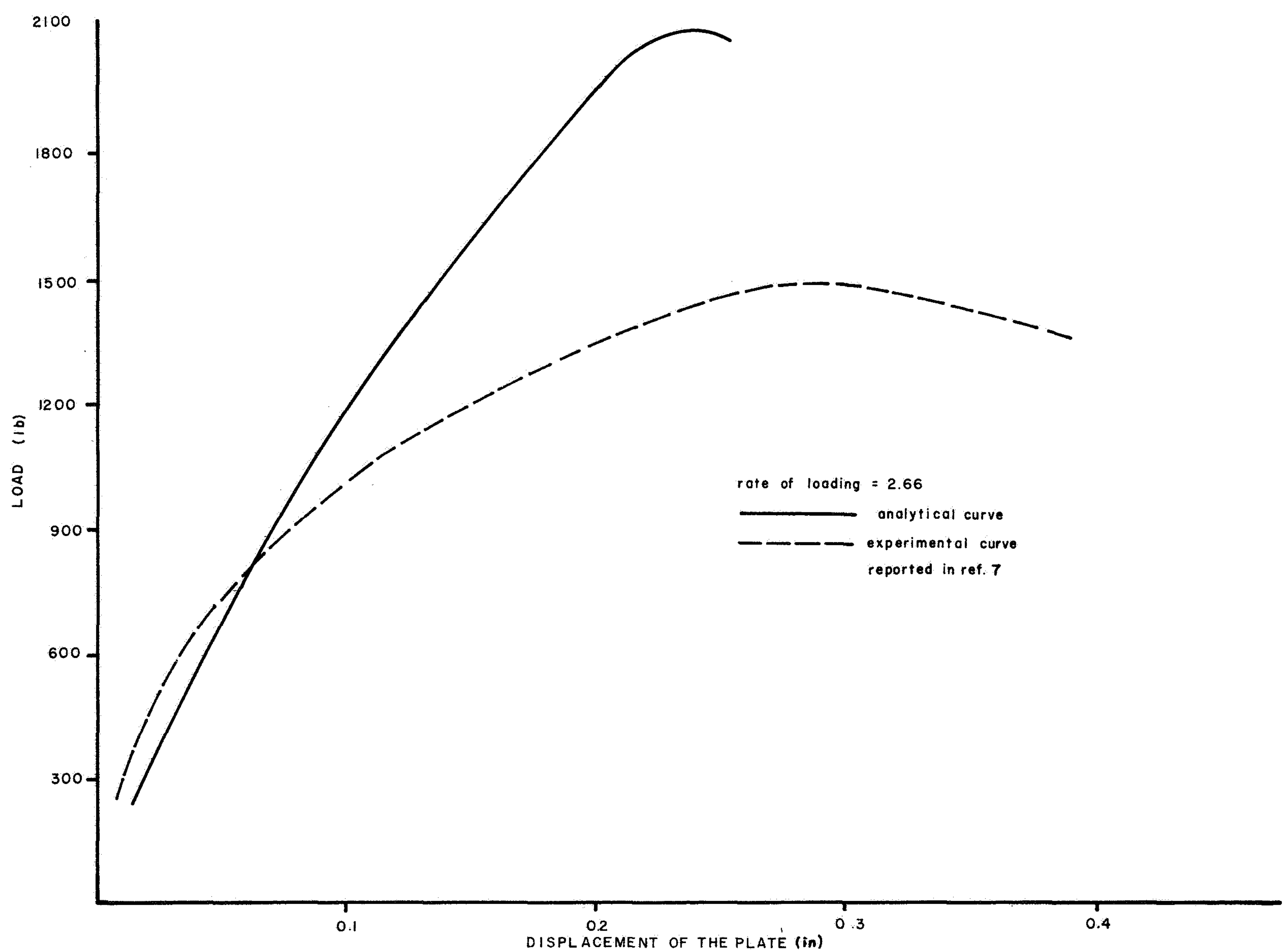


FIG. 5-9
LOAD VS. DISPLACEMENTS

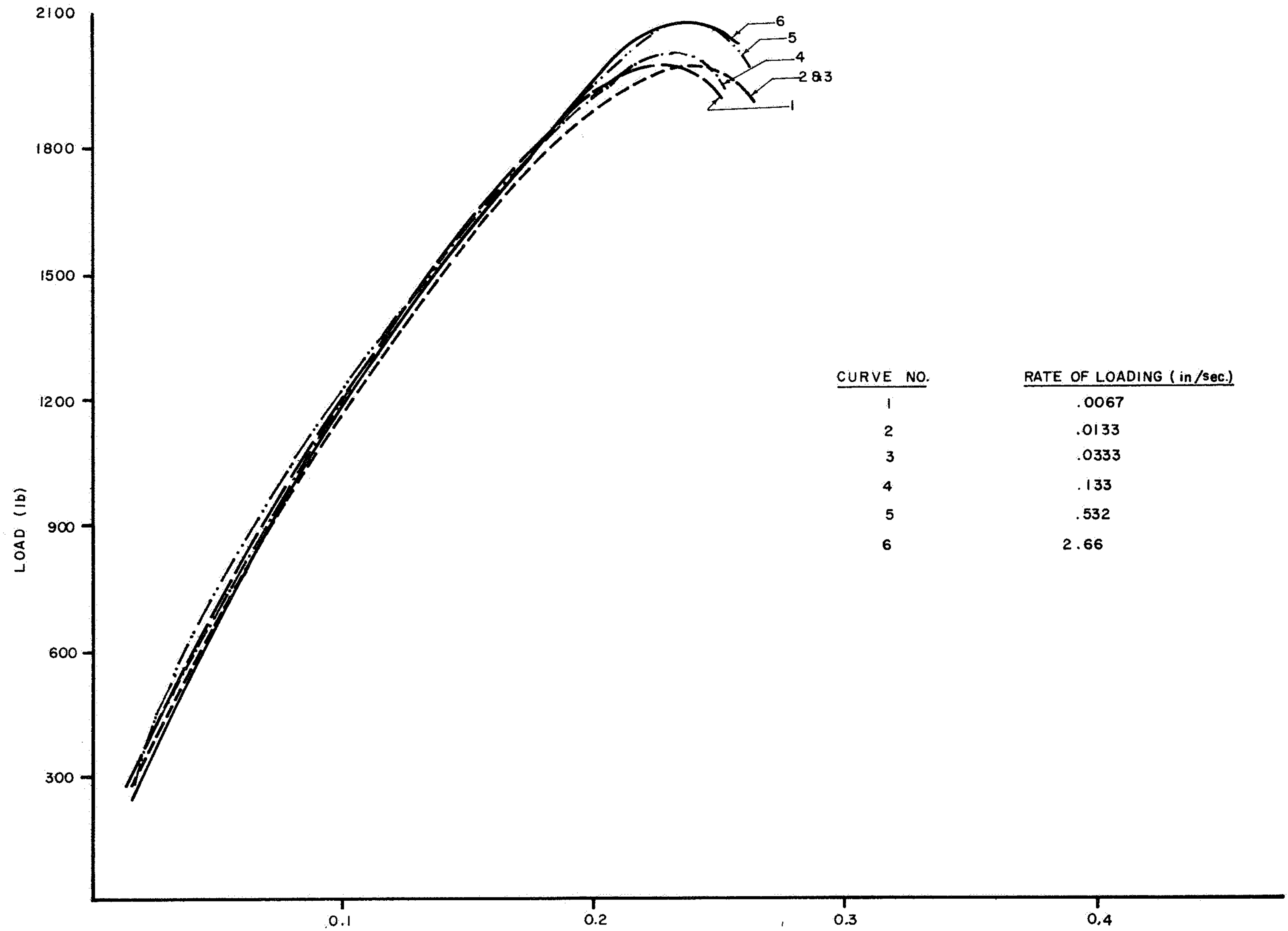


FIG. 5-10
ANALYTICAL LOAD-DISPLACEMENT CURVES

was obtained by treating the plate as a short retaining wall. For all the nodes on the surface $x = 0$, $y > 18''$ (cf. Fig. 5-1), displacements were set to zero. The rate of loading considered was 0.532 in./sec. The load-displacement curve for the above problem is shown in Fig. 5-8. From Fig. 5-8 it can be observed that this change in the boundary condition has a small effect for the rate of loading considered. It was not possible to take into account in the theoretical analysis other differences related to the load application as mentioned in Ref. (8).

In studying the effect of the rate of loading, solutions were obtained for higher rates of loading, keeping the same displacement increment as 0.0133 in. and decreasing the time interval. Figure 5-11 shows the displacement-load curves for the rates of loading of 0.0067, 0.0133, 0.0333, 0.133, 0.532, 2.66, 13.3, 26.6, and 39.9 in./sec, together with a portion of the curves for the rates of loading equivalent to 332.5 and 53.2 in./sec. It is apparent that there exists a difference in the resistance to movement between slow and high rates of loading. The difference increases with increase in displacement. The difference can be attributed to the increasing significance of inertia forces as rates of loading increase.

Table 5-1 shows the inertia forces at maximum load for different rates of loading. The node in the relaxation net for which the values in Table 5-1 were obtained has the coordinate (4.5, 9) that is 9 in. below the free surface and 4.5 in. away from the vertical surface of the plate. The inertia forces would be zero if displacements inside the region vary linearly with time, but the soil is a nonlinear material. If at a particular node, PX is the inertia force in the X direction and PY is the inertia force in the Y direction then,

TABLE 5-1

VARIATION OF INERTIA FORCE WITH ULTIMATE LOAD

Rate of Loading	Time Station at max. load	Horizontal Displacements			Vertical Displacements			Inertia Force in X direction	Inertia Force in Y direction
		u_k	u_{k-1}	u_{k-2}	v_k	u_{k-1}	v_{k-2}	$= \rho \ddot{u}$	$= \rho \ddot{v}$
in./sec	K	in.	in.	in.	in.	in.	in.	lb	lb
0.00667	19	0.2129	0.1999	0.1869	-.02360	-.02131	-.01908	0	2.25×10^{-9}
2.66	20	0.2262	0.2129	0.1999	-.02551	-.02336	-.02113	0.0018	0.00044
53.2	28	0.3230	0.3097	0.2966	-.02272	-.02678	-.02271	0.48	0.072
106.4	43	0.5061	0.4930	0.4800	-.1933	-.1820	-.1704	0.665	2.0

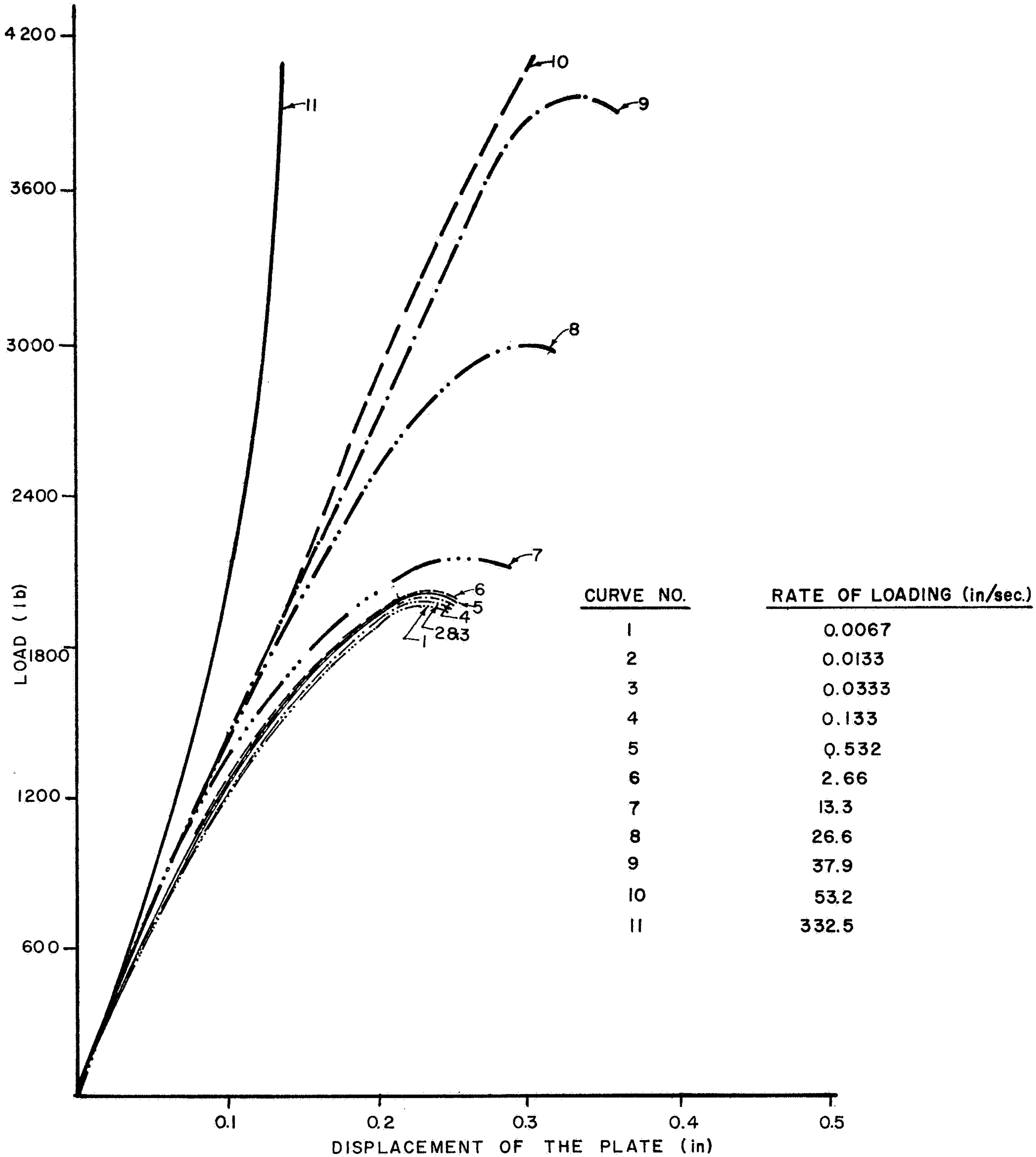


FIG. 5-11

ANALYTICAL LOAD-DISPLACEMENT CURVES

$$PX = \rho\, \ddot{u} = \rho\, \frac{(u_{i,j,k-2}\ u_{i,j,k-1} + u_{i,j,k})}{HT^2}$$

$$= \rho\, \frac{DX}{HT^2} \tag{5-2}$$

$$PY = \rho\, \ddot{v} = \rho\, \frac{(v_{i,j,k-2}\ v_{i,j,k-1}\ v_{i,j,k})}{HT^2}$$

$$= \rho\, \frac{DY}{HT^2} \tag{5-3}$$

PX and PY vanish if DX and DY become zero or if HT is very large. The inertia force could still be slight, despite nonlinearity, if HT is large, which is the case for the slow rates of loading. With the decrease in HT, that is, for higher rates of loading, the terms PX and PY would get larger. The inertia forces tend to bring the system back to its original position and hence, the displacements inside the region would be smaller for higher rates of loading if two rates are compared for the same boundary conditions. Table 5-2 shows displacements in the X direction of nodes on a vertical plane 4.5 in. away from the plate, the movement of the plate is 0.133 in. Table 5-3 shows the same information when the movement of the plate is 0.2393 in.

Figure 5-12 shows the load-displacement curves for rates of loading higher than 2.66 in./sec. It can be observed that a big difference exists between slow and high rates of loading, but, when high rates are compared together, the difference again becomes negligible. The terms slow and high as used in the text are arbitrary. For this particular problem, the following classification is adopted.

TABLE 5-2

HORIZONTAL DISPLACEMENTS (IN.) VS. RATE OF LOADING (IN./SEC)

OF NODES ON A VERTICAL PLANE 4.5 INCHES FROM THE PLATE

MOVEMENT OF PLATE = 0.133 IN.

Rate of Loading in./sec	Distance from Surface (in.)					
	3	6	9	12	15	18
2.66	0.1234	0.1233	0.1231	0.1227	0.1223	0.1218
13.3	0.1230	0.1226	0.1222	0.1216	0.1210	0.1205
53.2	0.1223	0.1219	0.1214	0.1209	0.1204	0.1197
106.4	0.1160	0.1163	0.1160	0.1149	0.1129	0.1088

TABLE 5-3

HORIZONTAL DISPLACEMENTS (IN.) VS. RATE OF LOADING (IN./SEC)

OF NODES ON A VERTICAL PLANE 4.5 INCHES FROM THE PLATE

MOVEMENT OF PLATE = 0.2393 IN.

Rate of Loading in./sec	Distance from Surface (in.)					
	3	6	9	12	15	18
2.66	0.2265	0.2264	0.2262	0.2259	0.2257	0.2255
13.3	0.2259	0.2257	0.2254	0.2251	0.2249	0.2249
26.6	0.2240	0.2233	0.2240	0.2215	0.2207	0.2205
53.2	0.2230	0.2214	0.2203	0.2192	0.2182	0.2175
106.4	0.2115	0.2111	0.2095	0.2073	0.2040	0.1964

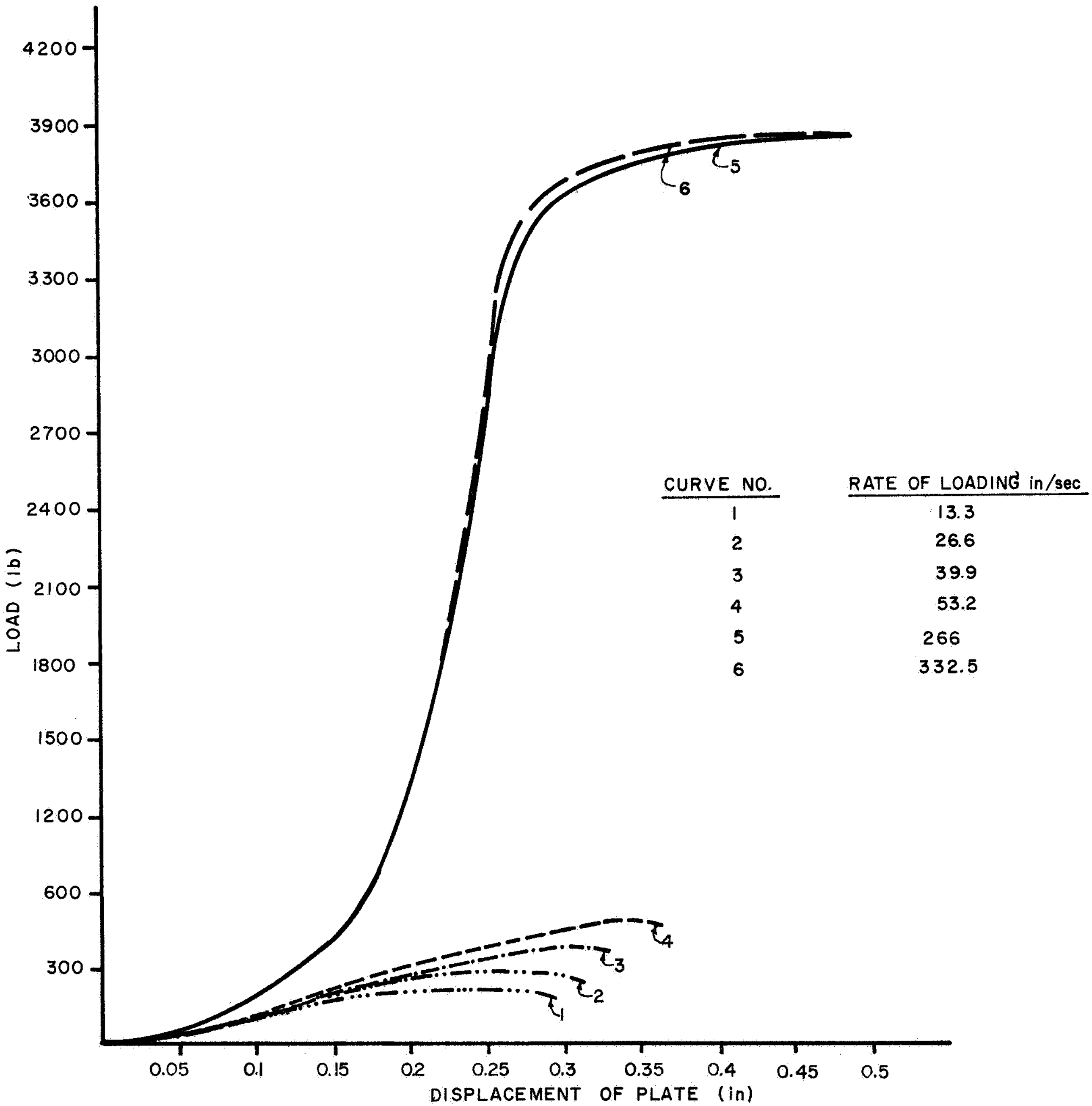


FIG. 5-12

ANALYTICAL LOAD DISPLACEMENT CURVES

Slow rates	0 - 2.60 in./sec
Intermediate rates	2.66 - 100 in./sec
High rates	> 100 in./sec

The above classification is based on the conclusion that for rates up to 2.66 in./sec the difference in ultimate resistance is small and for those greater than 100 in./sec, the difference in ultimate resistance also is small. Table 5-4 gives total load carried by the plate vs. the movement of the plate, for different rates of loading. The information in Table 5-4 was used in constructing Figs. 5-11 and 5-12. Figures 5-11 and 5-12 both indicate that for small displacements, less than 0.1 in., the load carried by the plate differs slightly between slow, intermediate and high rates of loading.

Figure 5-13 shows the ultimate load vs. the rate of loading. The semi-flat part of the curves is for high rates of loading, greater than 100 in./sec. The shape of the curve is similar to that of the so-called error function (4) except that the initial portion does not start from zero and the middle portion is steeper than that of the error function. In order to obtain a mathematical model to describe the curve, the curve is redrawn again in Fig. 5-14 with non-dimensional coordinates.

The vertical scale contains nondimensional quantities which were obtained by dividing the ultimate load by the maximum ultimate load. The maximum ultimate load is considered that which corresponds to a rate of loading of 200 in./sec, for which the ultimate load is 35,800 lbs.

Values on the horizontal scale are divided by 200 and multiplied by 3 so that the relation would be $\frac{P_{ult}}{P_{ult_{max}}}$ vs Z where Z is equal to $\frac{3R}{200}$.

The error function of 3 is very close to one which is the maximum value on the vertical scale. Based on Fig. 5-14 the general relation will be:

TABLE 5-4

TOTAL LOAD (LB) VS. MOVEMENT OF THE PLATE (IN.)

Movement of Plate, (in.)	Rate, in./sec 332.5	266	53.2	39.9	26.6	13.3	2.66	0.532	0.133	0.0333	0.0133	0.00667
0.0266	511.8	511.8	434.2	434.8	439.2	437	471.9	434	432.9	432.8	432.8	432.8
0.0399	769.9	770	587.6	591.3	600.5	603.7	620.9	578.6	578.7	578.8	578.8	578.9
0.0532	1035	1035	741.8	748.2	761	768.2	738.6	720.5	715.7	715.8	715.8	721.3
0.0665	1311	1311	897.6	906.2	912.1	928.9	851.2	848.2	846.7	847	847	849
0.0798	1606	1606	1056	1066	1081	1084	967.1	979.1	979.8	980.4	980.4	972.5
0.0931	1926	1927	1217	1228	1241	1230	1097	1104	1102	1105	1105	1102
0.1064	2283	2284	1382	1392	1402	1367	1225	1221	1223	1223	1223	1222
0.1197	2693	2695	1551	1561	1563	1494	1347	1334	1335	1335	1335	1336
0.133	3181	3184	1726	1732	1724	1610	1458	1452	1453	1454	1454	1455
0.1463	3790	3793	1907	1908	1884	1714	1565	1558	1552	1561	1561	1561
0.1596	4602	4609	2096	2089	2041	1805	1664	1660	1661	1652	1652	1650
0.1729	5836	5854	2292	2274	2195	1884	1757	1744	1746	1741	1741	1739
0.1862	8436	8436	2497	2464	2342	1953	1830	1823	1826	1824	1824	1824
0.1995	9957	9951	2711	2658	2482	2012	1900	1886	1888	1890	1890	1897
0.2128	12550	12530	2935	2853	2608	2061	1957	1938	1941	1939	1939	1944
0.2260	16690	16650	3143	3024	2701	2089	1985	1969	1971	1966	1966	1977
0.2393	20050	19940	3387	3217	2800	2123	1991	1971	1974	1971	1971	1969
0.2526	25560	25220	3639	3402	2880	2140	1950	1888	1881	1886	1886	
0.2659	32410	31750	3898	3574	2936	2143						
0.2792	33710	33670	4157	3724	2964	2127						
0.2925	34010	33950	4409	3842	2962							
0.3058	34250	34190	4641	3919								
0.3191	34460	34380	4836	3948								
0.3324	34620	34530	4973	3915								
0.3457	34750	34660	5022									
0.3590	34860	34760	4950									
0.3723	34940	34840										
0.3856	35010	34900										
0.3989	35060	34960										

TABLE 5-4 CONTINUED

TOTAL LOAD (LB) VS. MOVEMENT OF THE PLATE (IN.)

Movement of Plate, (in.)	Rate in./sec											
	332.5	266	53.2	39.9	26.6	13.3	2.66	0.532	0.133	0.0333	0.0133	0.00667
0.4122	35100	35000										
0.4255	35130	35040										
0.4388	35160	35070										
0.4521	35180	35100										
0.4654	35210	35120										
0.4787	35230	35150										
0.4920	35250	35170										
0.5053	35270	35200										
0.5186	35290	35220										
0.5319	35320	35250										
0.5452	35340	35270										
0.5585	35370	35300										

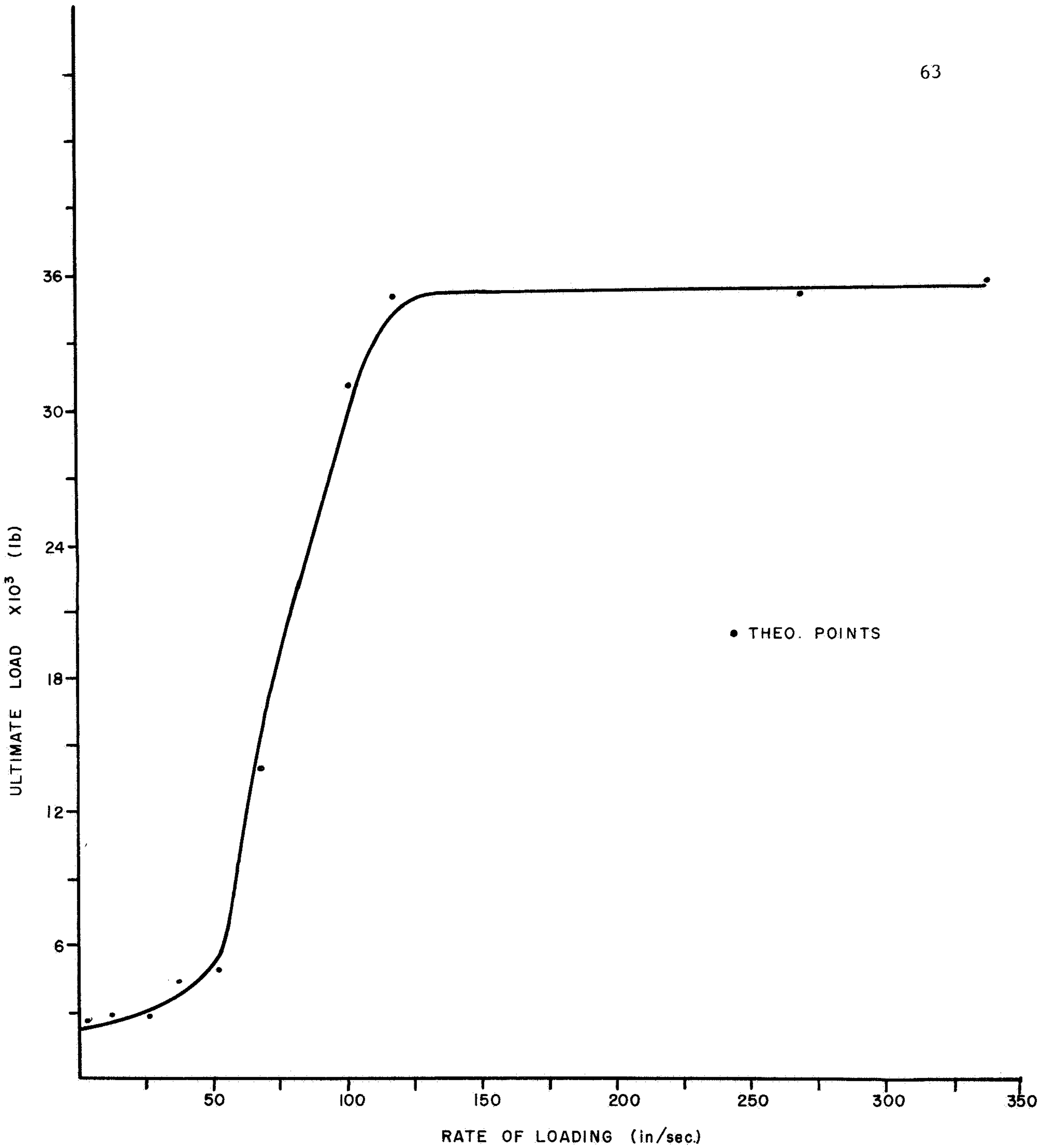


FIG. 5-13

ULTIMATE LOAD VS. RATE OF LOADING

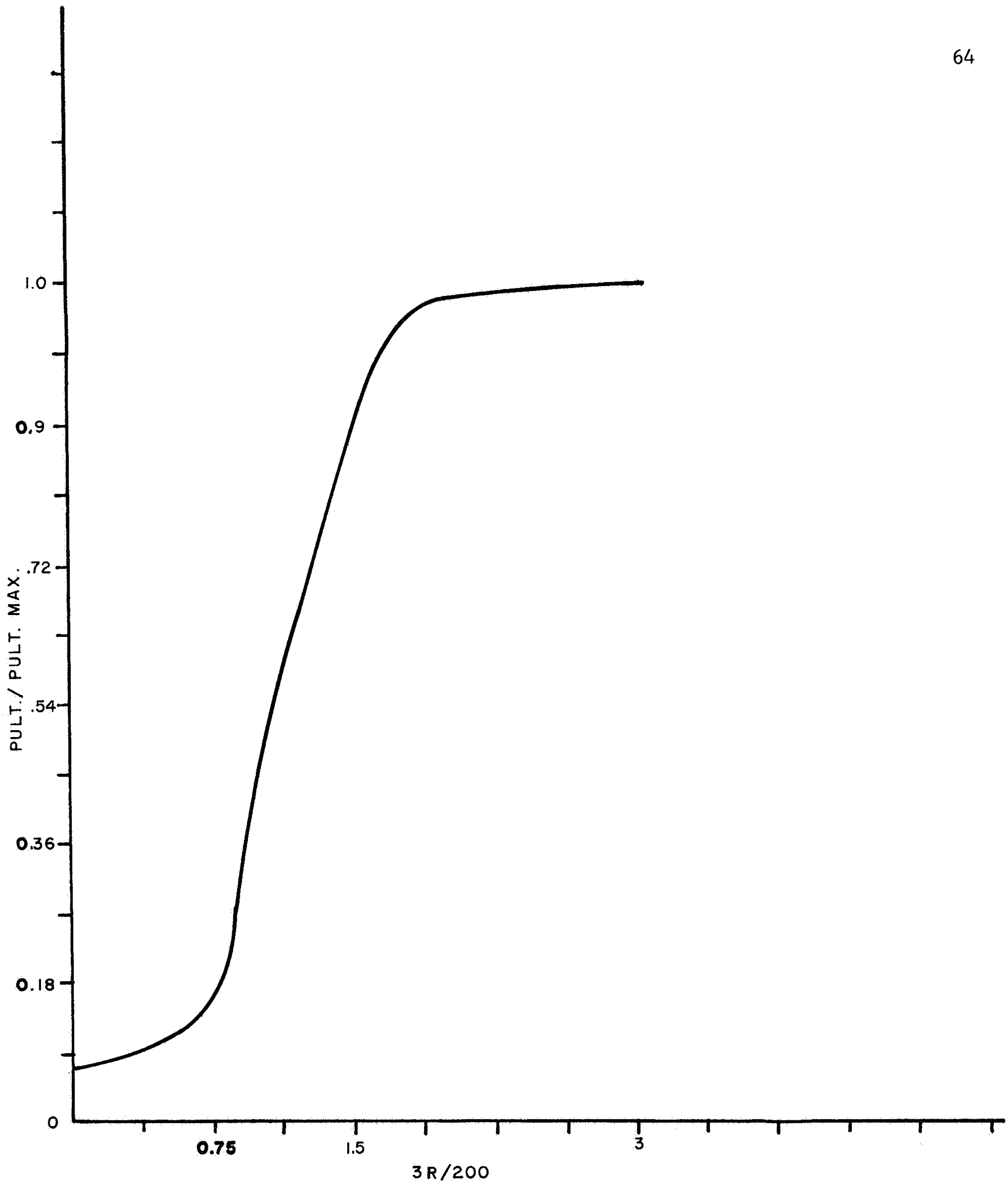


FIG. 5-14

ULTIMATE LOAD VS. RATE OF LOADING

IN NONDIMENSIONAL FORM

$$\frac{P_{ult}}{P_{ult_{max}}} = \frac{2}{\sqrt{\pi}} \int_0^{Z-C} e^{-\alpha^2} \, d\alpha \tag{5-4}$$

in which Z is greater than C.

The value of C is found by trial and error to be 0.6 for $Z > 0.6$ and to be equal to 0.92 Z for $0 < Z \leq 0.6$. Another constant should be added since the curve does not start from zero values. So that the relations take the form:

for $0 \leq Z \leq 0.6$

$$\frac{P_{ult}}{P_{ult_{max}}} = \frac{2}{\sqrt{\pi}} \int_0^{0.08Z} e^{-\alpha^2} \, d\alpha + 0.056 \tag{5-5}$$

and for $0\text{-}6 < Z < 3.0$

$$\frac{P_{ult}}{P_{ult_{max}}} = \frac{2}{\sqrt{\pi}} \int_0^{Z - 0.6} e^{-\alpha^2} \, d\alpha \tag{5-6}$$

where

$Z = \frac{3R}{200}$

R = rate of loading in./sec.

The integrals in Eqs. 5-5 and 5-6 could be evaluated numerically or obtained from mathematical tables (11). The program listed in Appendix D evaluates the integral.

Average Earth Pressure Modulus

The average earth pressure modulus as used in this text is defined as:

$$K = \frac{P}{A\ d}$$

P total resistance to plate movement

K is the average earth pressure modulus in $lb/in.^2/in.$

A the area of the plate ($216\ in.^2$)

d movement of the plate (in.)

Figure 5-15 shows the variation of K with movement of the plate. For low movement, less than 0.05 in., the behavior for all the rates is the same. That is, there is a decrease in K with an increase in d. For larger movement of the plate, the curve for the rate of 2.66 in./sec continues to show the decrease in K for an increase in d. For the rate of loading of 2.66 in./sec the inertia force is negligible and behavior corresponds to that of the so-called static case. For intermediate, 53.2 in./sec, and high, 332.5 in./sec, rates, the curve shows an increase of K with increased d for movements greater than 0.05 in.

Effects of Rate of Loading on Distribution of Horizontal Stress

The plane chosen for discussion here is 18 in. below the surface. The distribution of horizontal stress on nodes along this horizontal plane will be considered.

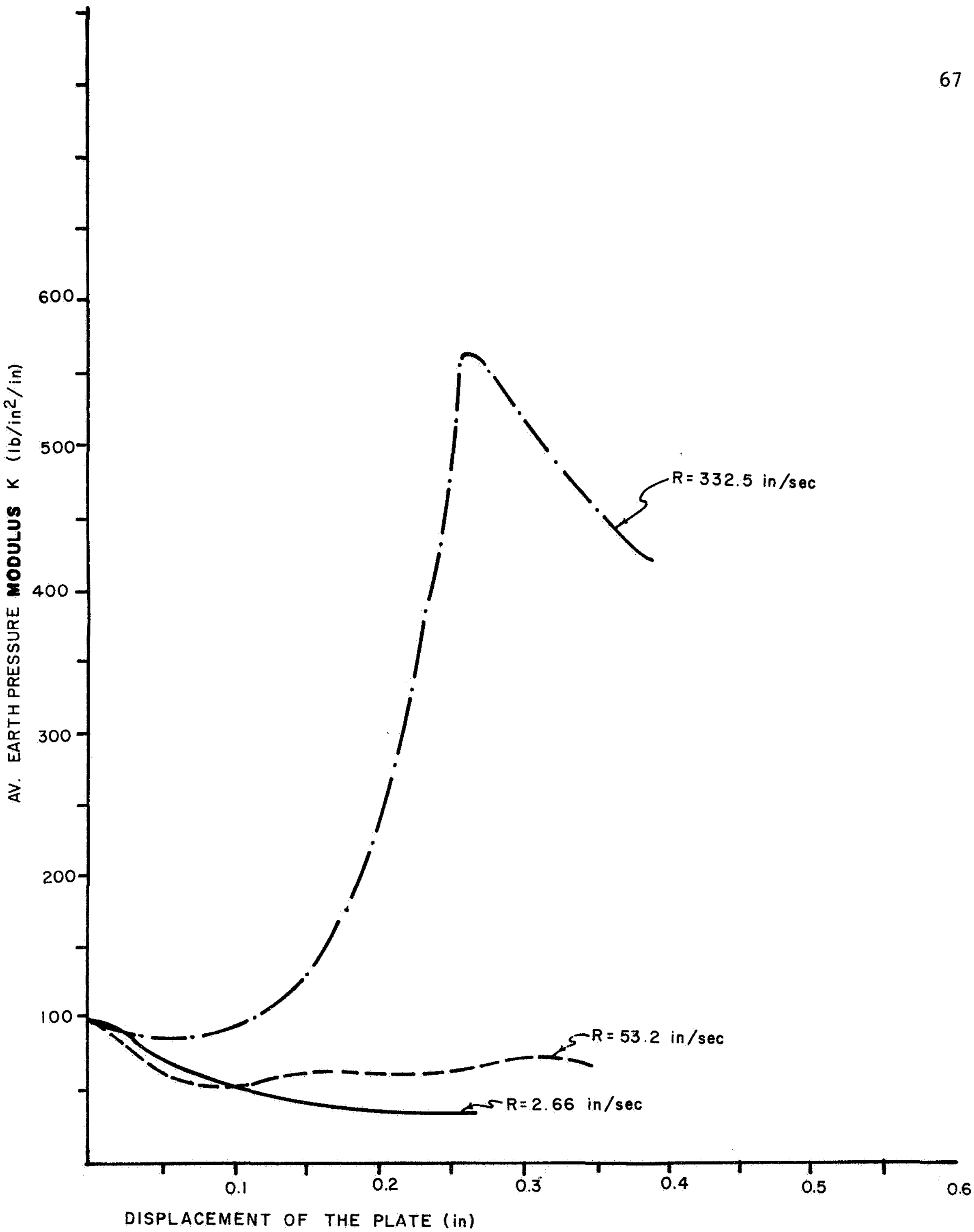


FIG. 5-15

AVERAGE EARTH PRESSURE MODULUS

VS. PLATE MOVEMENT

In Tables 5-5, 5-6, and 5-7 the horizontal stresses are listed at equally spaced nodes on a plane 18 in. below the soil surface. The distance of horizontal movements for the plates in Tables 5-5, 5-6, and 5-7 are 0.0665, 0.133, and 0.2128 in. respectively. Data from the three Tables are plotted in Figs. 5-16, 5-17, and 5-18. It can be seen in these figures that the rate of loading has a marked effect on the distribution of stresses for nodes close to the plate. This effect decreases with an increase in distance from the plate.

The three Figures, 5-16, 5-17, and 5-18, show that the rate of loading has a marked effect on the distribution of stress as well as on the values of stresses for nodes situated within a distance equal to approximately the depth of the plate (18 in.) which is the bearing surface. Stresses increase with an increase in the rate of loading when slow rates are compared with intermediate and high rates or when intermediate rates are compared with high rates. The other observation is that when slow rates are compared together the distribution is practically the same. The same observation applies also when high rates are compared together.

Distribution of Stresses on the Plate

Figures 5-19, 5-20, and 5-21 show the distribution of stresses on the plate for different rates of loading for three values of plate movements. Nonlinearity of the curves increase with increased rate of loading. For the rate of loading of 2.66 in./sec, the movement of the plate of 0.2128 is close to that which corresponds to maximum load and the distribution is rather uniform and the nodes along the plate cannot sustain more stresses even though the displacements increase with time. The resistance of the neighboring nodes is greater. This is shown in Fig. 5-19 by observing the stress distribution

TABLE 5-5

VARIATION OF HORIZONTAL STRESS (PSI) WITH DISTANCE (IN.)

OF NODES ON A HORIZONTAL PLANE 18 INCHES FROM THE SURFACE

MOVEMENT OF THE PLATE = 0.0665 IN.

Rate of Loading	Distance from the Plate (in.)											
in./sec	0	4.5	9	13.5	18	22.5	27	31.5	36	40.5	45	49.5
1.33	5.030	4.835	4.782	4.587	4.300	3.963	3.618	3.297	3.025	2.815	2.660	2.537
2.66	5.084	4.898	4.845	4.634	4.327	3.969	3.604	3.270	2.791	2.777	2.629	2.512
13.3	5.779	5.537	5.322	4.908	4.435	3.979	3.572	3.228	2.959	2.768	2.648	2.571
106.4	8.897	7.931	7.177	6.308	5.343	4.422	3.647	3.056	2.643	2.380	2.237	2.183
332.5	8.856	7.929	7.201	6.317	5.343	4.418	3.643	3.055	2.644	2.382	2.239	2.177

TABLE 5-6

VARIATION OF HORIZONTAL STRESS (PSI) WITH DISTANCE (IN.)

OF NODES ON A HORIZONTAL PLANE 18 INCHES FROM THE SURFACE

MOVEMENT OF THE PLATE = 0.133 IN.

Rate of Loading in./sec	Distance from the Plate (in.)											
	0	4.5	9	13.5	18	22.5	27	31.5	36	40.5	45	49.5
.133	8.135	7.891	8.188	8.126	7.86	7.473	7.035	6.597	6.194	5.84	5.524	5.204
2.66	8.215	7.969	8.262	8.185	7.9	7.494	7.038	6.585	6.171	5.808	5.486	5.161
13.3	9.361	9.17	9.428	9.145	8.587	7.914	7.241	6.645	6.164	5.806	5.541	5.30
39.9	10.26	9.895	9.740	9.309	8.799	8.282	7.800	7.387	7.072	6.865	6.755	6.695
106.4	26.94	21.33	17.42	14.51	11.83	9.544	7.743	6.412	5.492	4.910	4.592	4.484
332.5	26.20	21.34	17.74	14.62	11.82	9.495	7.899	6.391	5.493	4.922	4.598	4.441

TABLE 5-7

VARIATION OF HORIZONTAL STRESS (PSI) WITH DISTANCE (IN.)

OF NODES ON A HORIZONTAL PLANE 18 INCHES FROM THE SURFACE

MOVEMENT OF THE PLATE = 0.2128 IN.

Rate of Loading in./sec	Distance from the Plate (in.)											
	0	4.5	9	13.5	18	22.5	27	31.5	36	40.5	45	49.5
.133	10.30	9.958	11.08	11.71	11.94	11.8	11.37	10.8	10.17	9.526	8.856	8.108
2.66	10.47	10.13	11.27	11.85	12.01	11.8	11.34	10.75	10.12	9.478	8.815	8.074
39.9	17.62	17.15	17.59	16.67	15.45	14.32	13.38	12.69	12.22	11.97	11.85	11.68
53.2	17.82	17.35	17.53	16.65	15.77	14.82	14.07	13.38	12.95	12.72	12.64	12.56
266	171.4	157.6	72.58	38.21	24.83	17.86	13.69	11.04	9.333	8.281	7.687	7.407
332.5	171.3	157.8	73.79	38.33	24.82	17.83	13.67	11.03	9.332	8.283	7.689	7.395

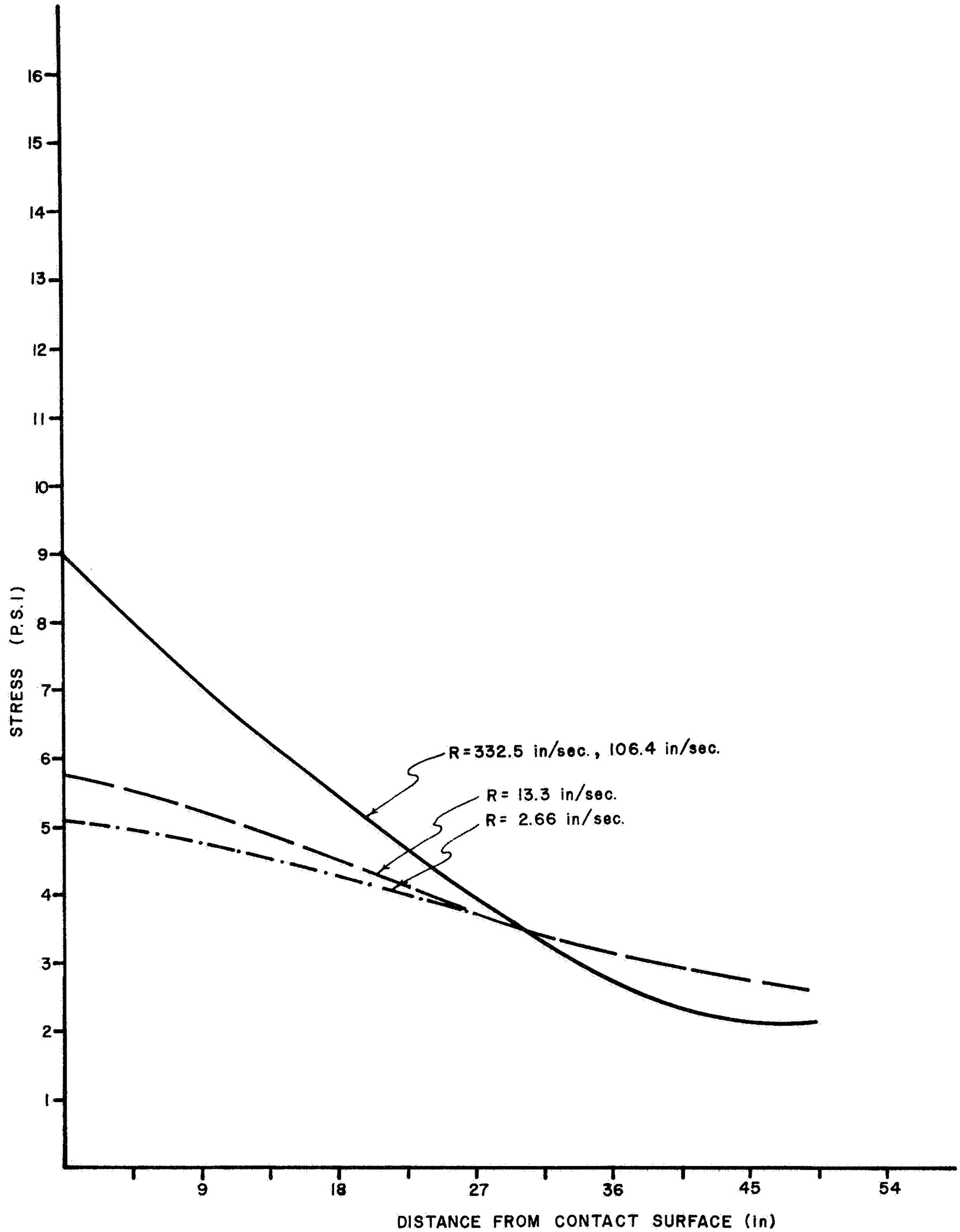


FIG. 5-16

HORIZONTAL STRESS VS. DISTANCE

MOVEMENT OF THE PLATE = 0.0665 IN.

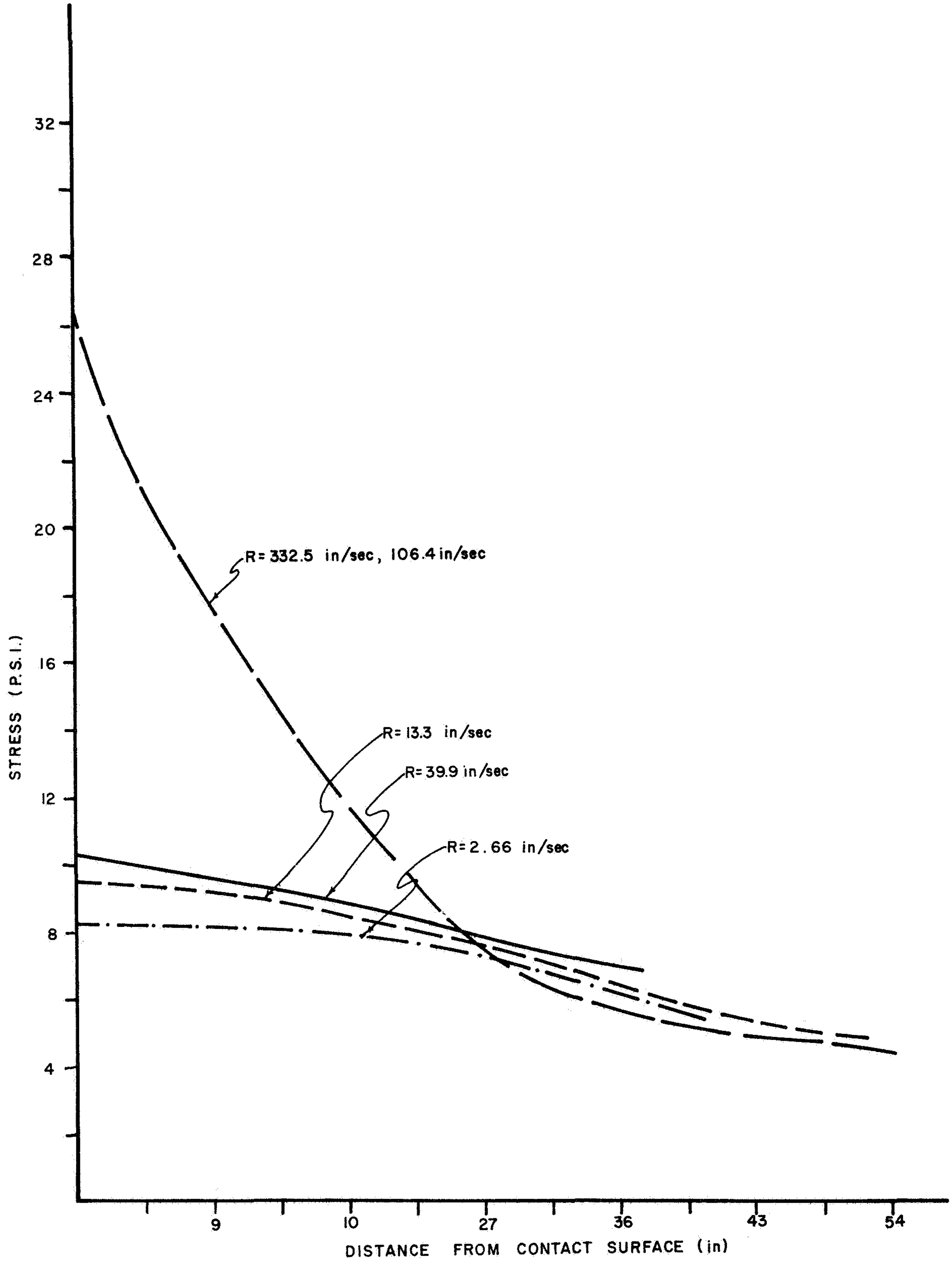


FIG. 5-17

HORIZONTAL STRESS VS. DISTANCE
MOVEMENT OF THE PLATE = 0.133 IN.

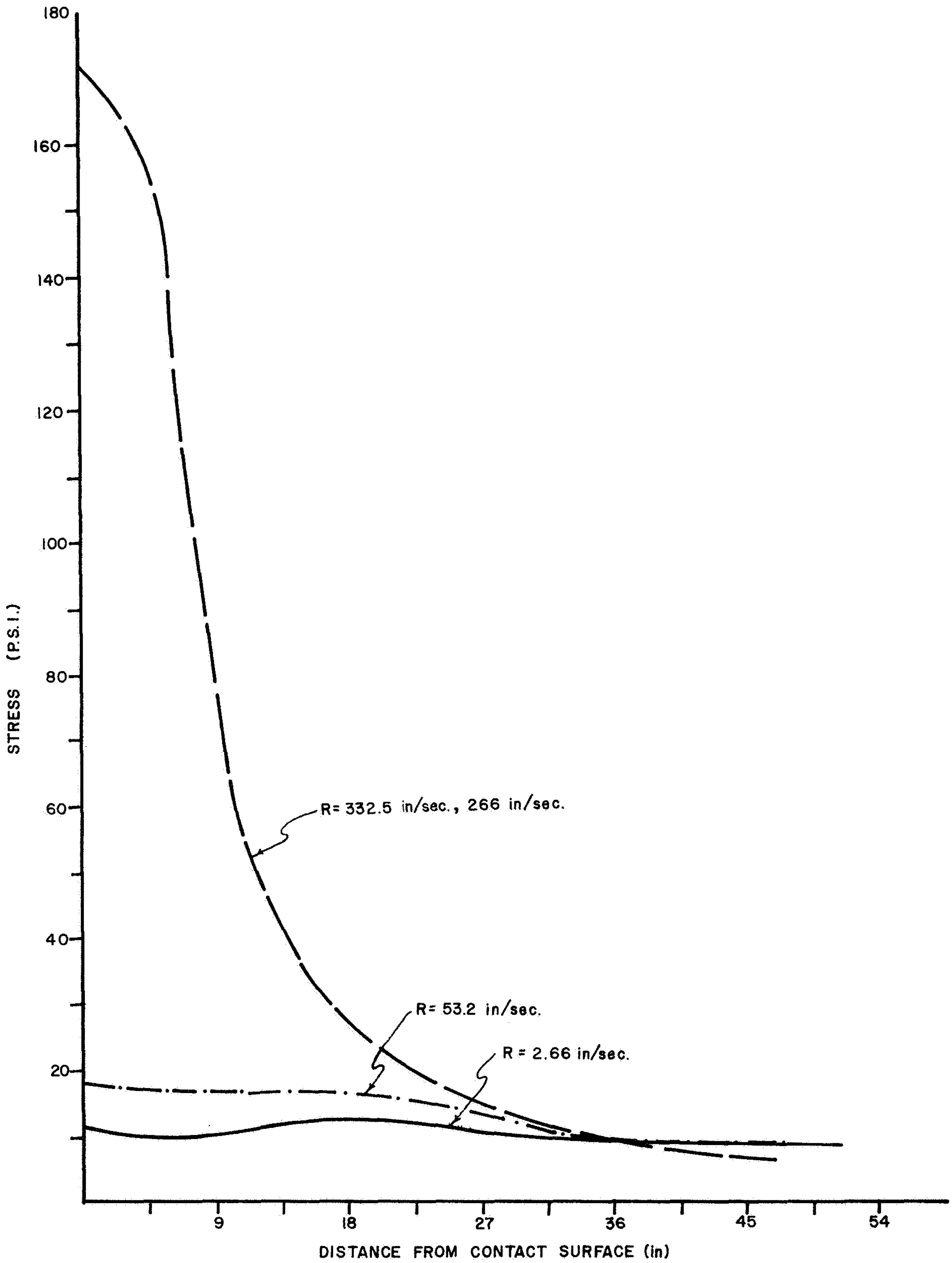


FIG. 5-18

HORIZONTAL STRESS VS. DISTANCE
MOVEMENT OF THE PLATE = 0.2128 IN.

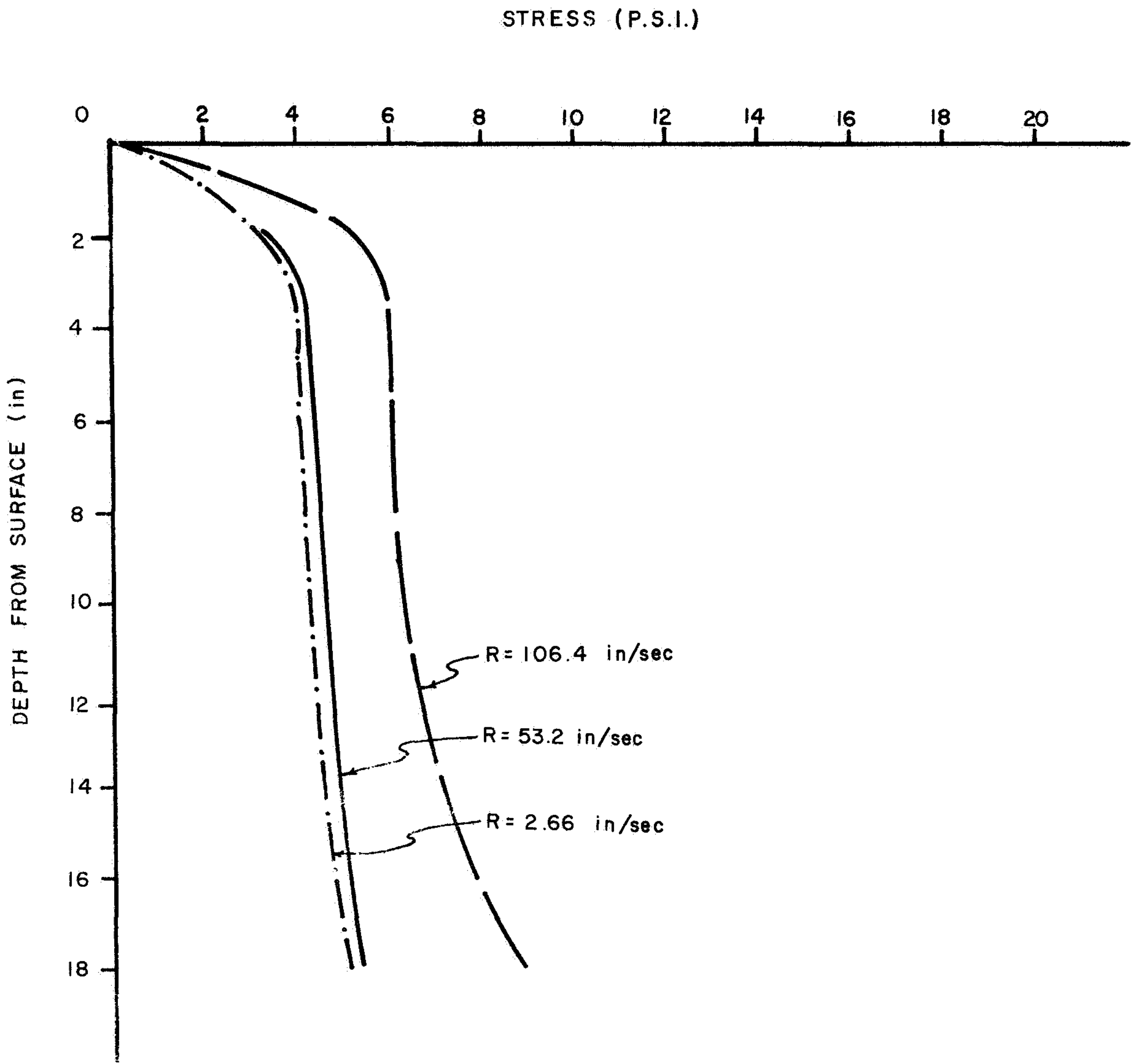


FIG. 5-19

DISTRIBUTION OF HORIZONTAL STRESS ON CONTACT SURFACE

MOVEMENT OF THE PLATE = 0.0665 IN.

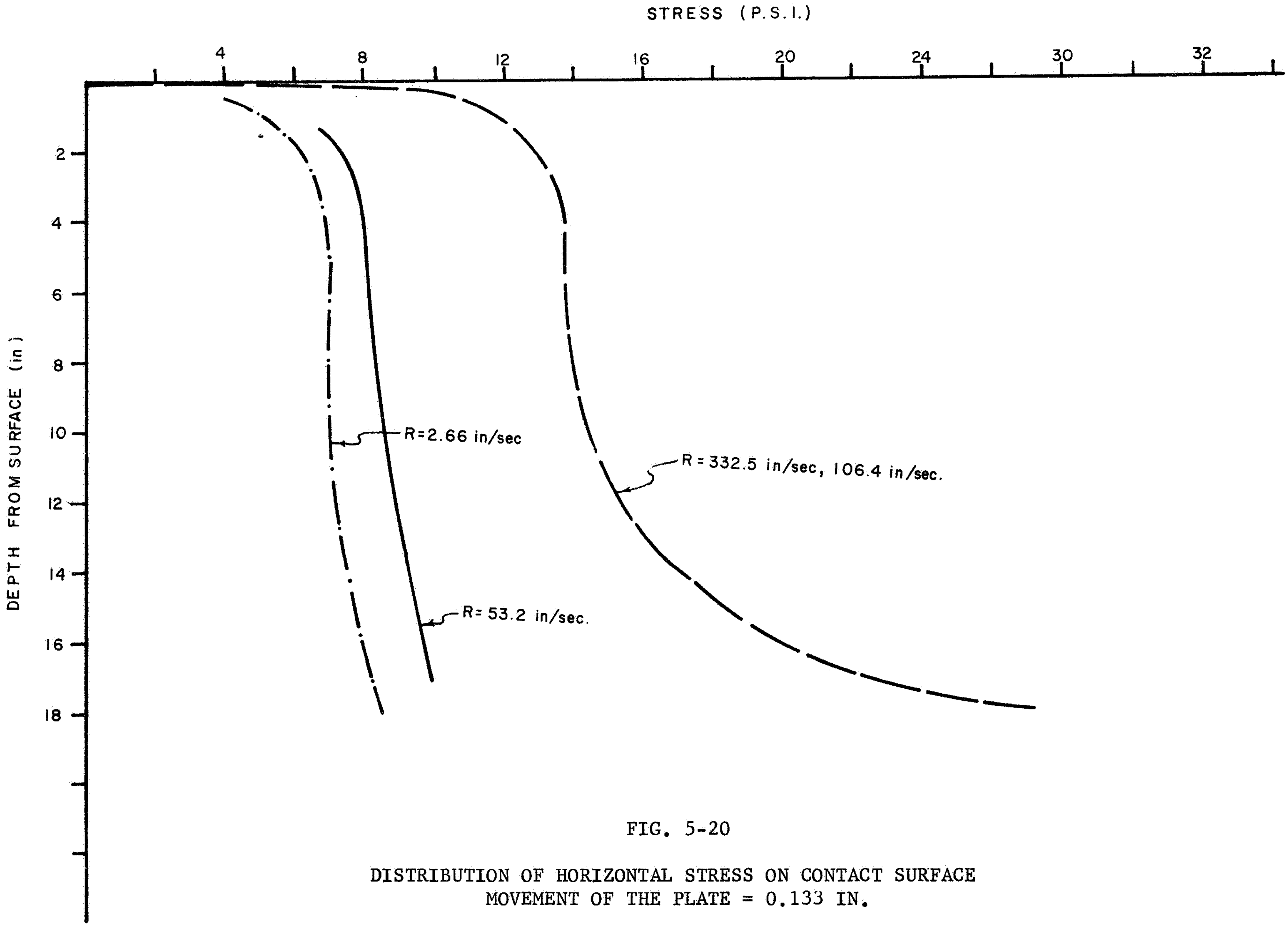


FIG. 5-20

DISTRIBUTION OF HORIZONTAL STRESS ON CONTACT SURFACE
MOVEMENT OF THE PLATE = 0.133 IN.

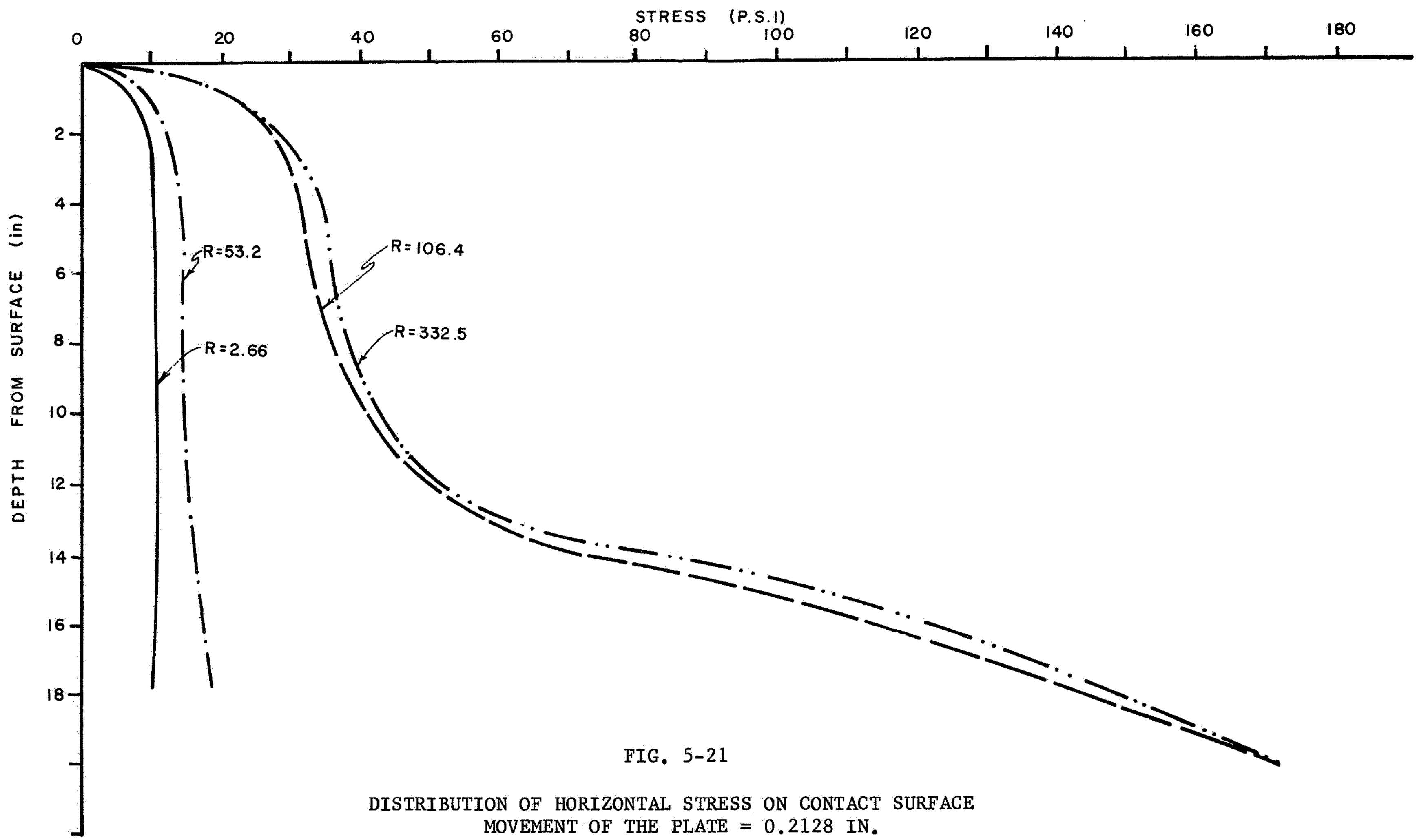


FIG. 5-21

DISTRIBUTION OF HORIZONTAL STRESS ON CONTACT SURFACE
MOVEMENT OF THE PLATE = 0.2128 IN.

curve for the rate of loading of 2.66. For higher rates of loading the same situation will develop when the movement of the plate is higher than 0.2128 in. The latter observation can be seen by examining the complete results from output of different rates of loading.

Distributions of Stresses and Displacements in the Soil Mass

Figures 5-22 through 5-27 show the contour lines for horizontal stresses and horizontal displacements for the region bounded by the plate. Three rates of loading corresponding to 2.66, 106.4 and 266 in./sec were considered. No general conclusion could be drawn from the contours shown since such contours depend on the rate of loading and the displacement of the plate. An argument related to these contours should be built on large numbers of such contours. The shape of such contours, however, shows that the distribution of stresses and displacements is compatible with the boundary conditions.

Conclusions

The two quantities E' and ν', which are analogous to the modulus of elasticity E and Poisson's ratio ν, have been found to represent the soil properties provided that such quantities be allowed to vary with strain level. The resistance of a 102 pcf density sand to penetration of an 18 x 12-in. plate at constant velocity has been investigated for different rates of loading. From the force displacement histories several conclusions have been drawn:

1. For rates of loading up to 2.66 in./sec, no significant change occurs in the resistance and in the state of stress and deformation in the medium.

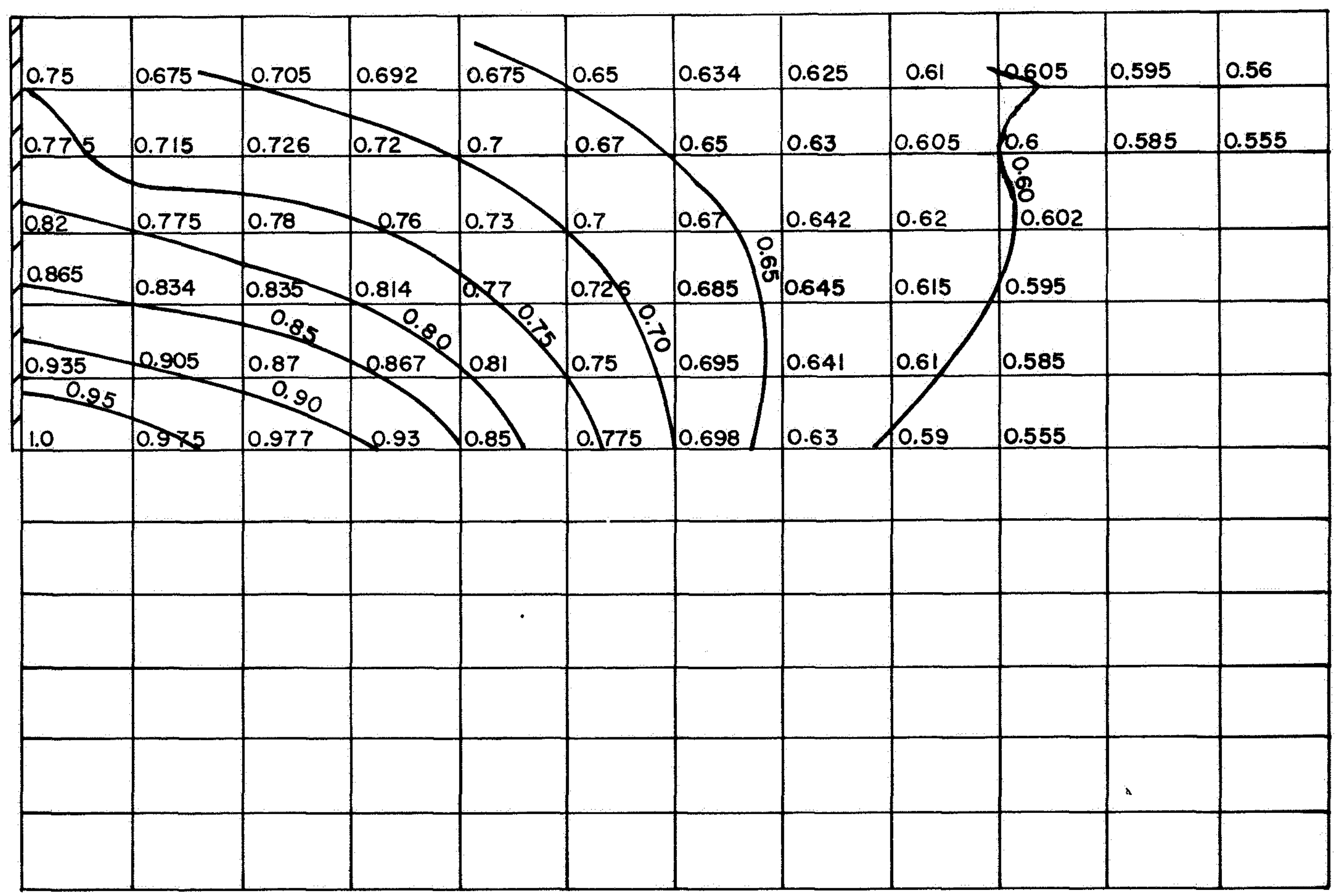


FIG. 5-22

CONTOURS OF HORIZONTAL STRESSES/MAXIMUM STRESS FOR THE REGION BOUNDED BY THE PLATE. RATE OF LOADING = 2.66 IN./SEC DISPLACEMENT OF PLATE = 0.1064 IN.

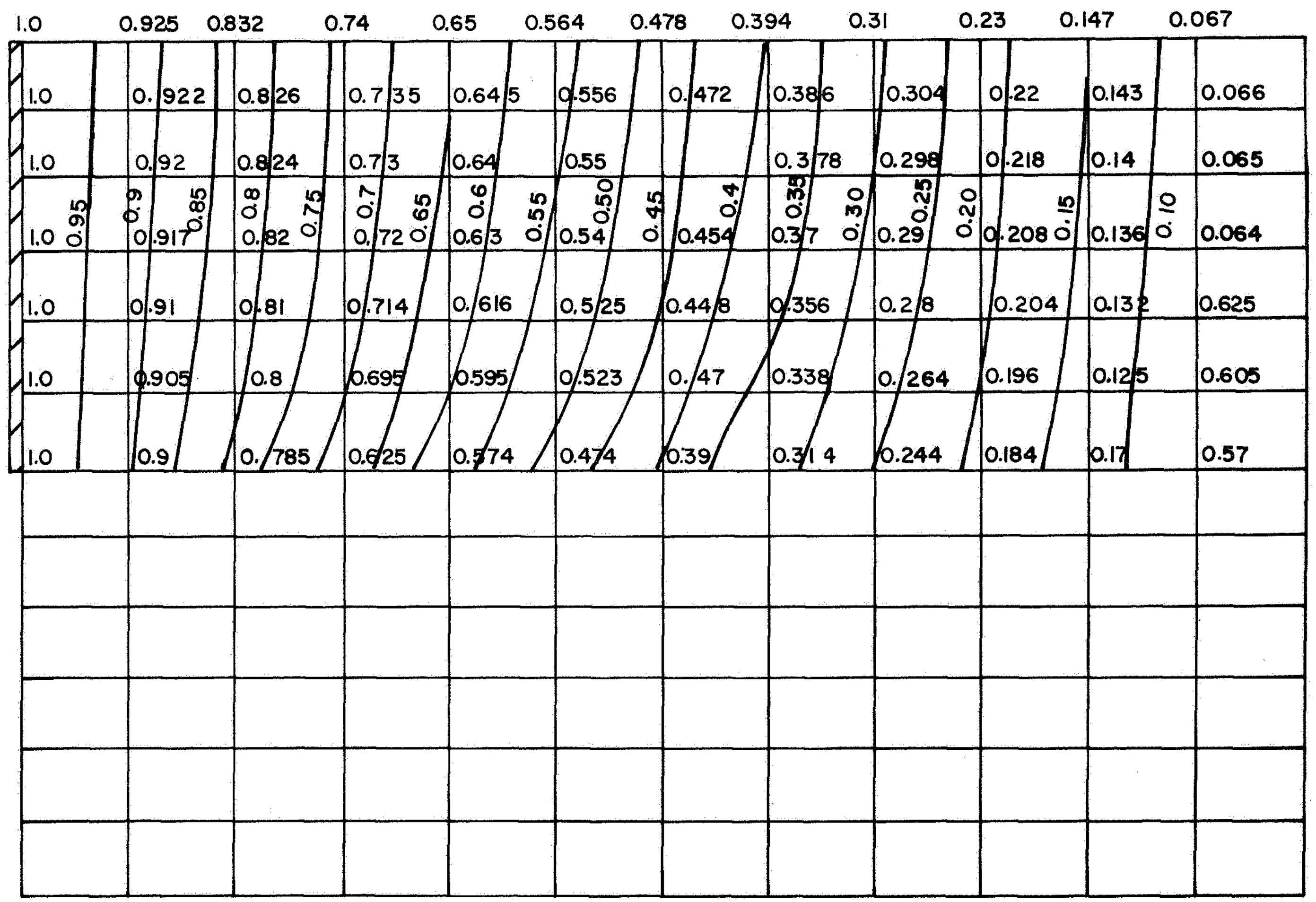


FIG. 5-23

CONTOURS OF HORIZONTAL DISPLACEMENT/PLATE DISPLACEMENT FOR THE REGION BOUNDED BY THE PLATE. RATE OF LOADING = 2.66 IN./SEC DISPLACEMENT OF THE PLATE = 0.1064 IN.

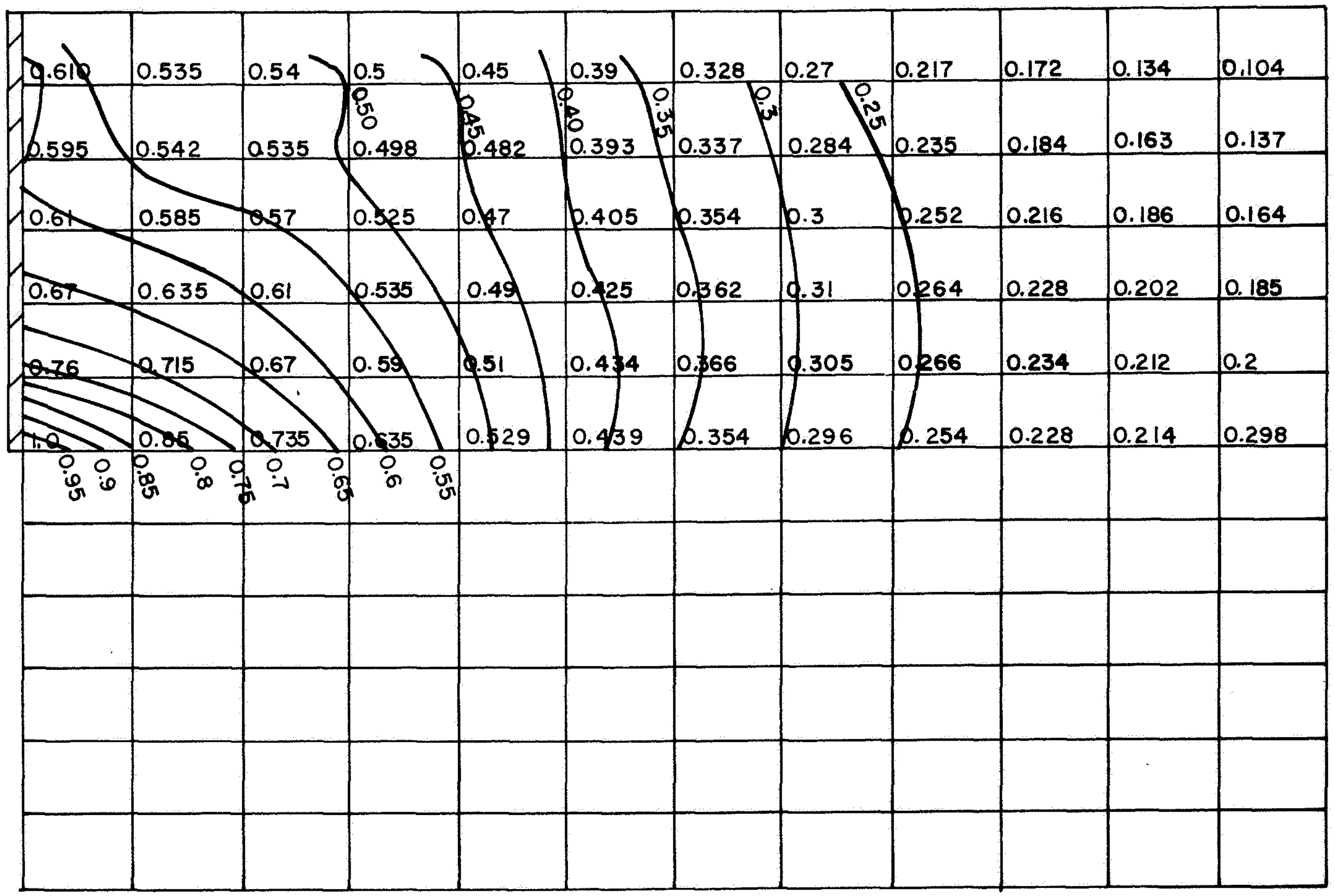
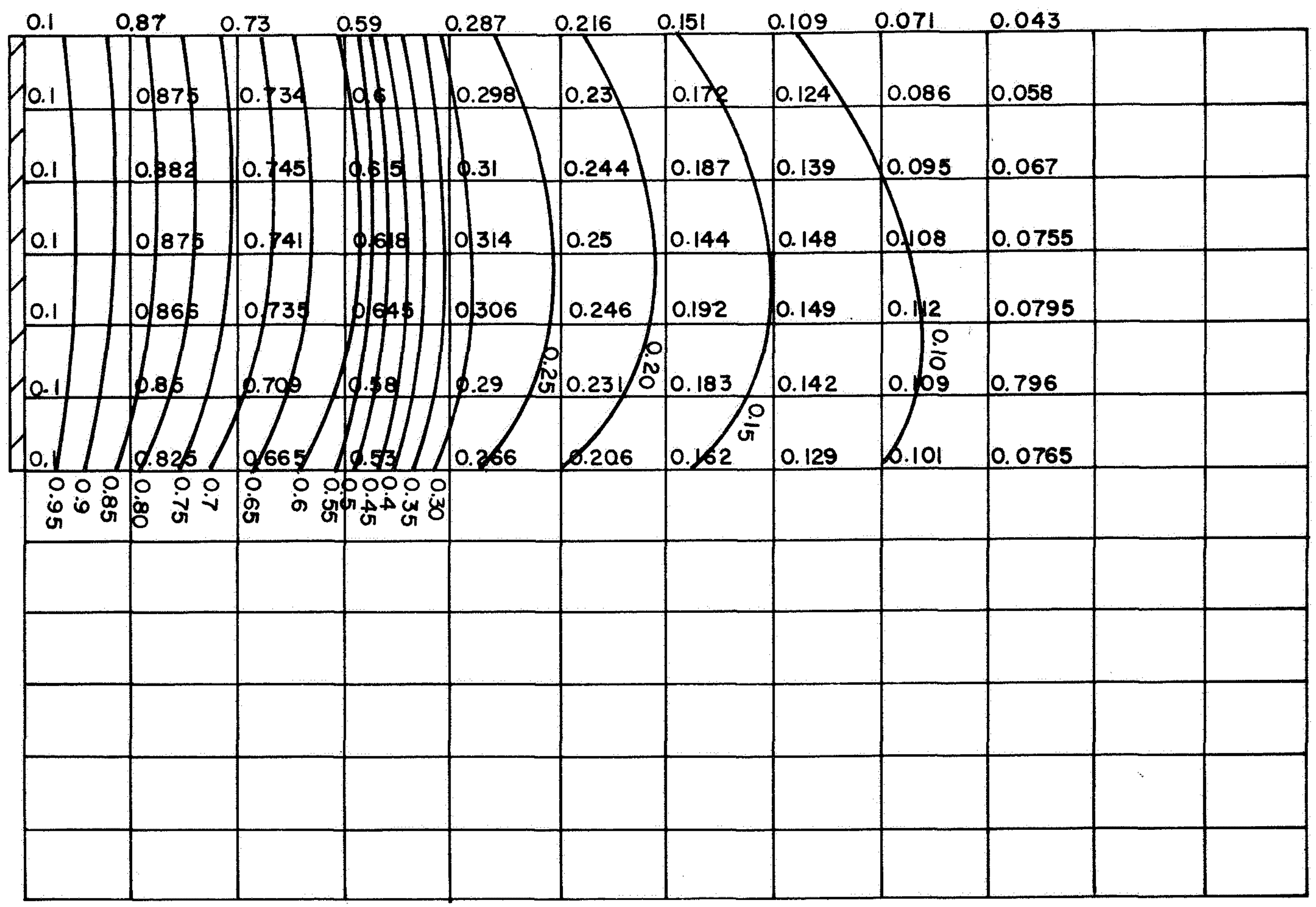


FIG. 5-24

CONTOURS OF HORIZONTAL STRESSES/MAXIMUM STRESS FOR THE REGION BOUNDED BY THE PLATE. RATE OF LOADING = 106.4 IN./SEC DISPLACEMENT OF THE PLATE = 0.1064 IN.



·FIG. 5-25

CONTOURS OF HORIZONTAL DISPLACEMENT/PLATE DISPLACEMENT FOR THE REGION BOUNDED BY THE PLATE. RATE OF LOADING = 106.4 IN./SEC PLATE MOVEMENT = 0.1064 IN.

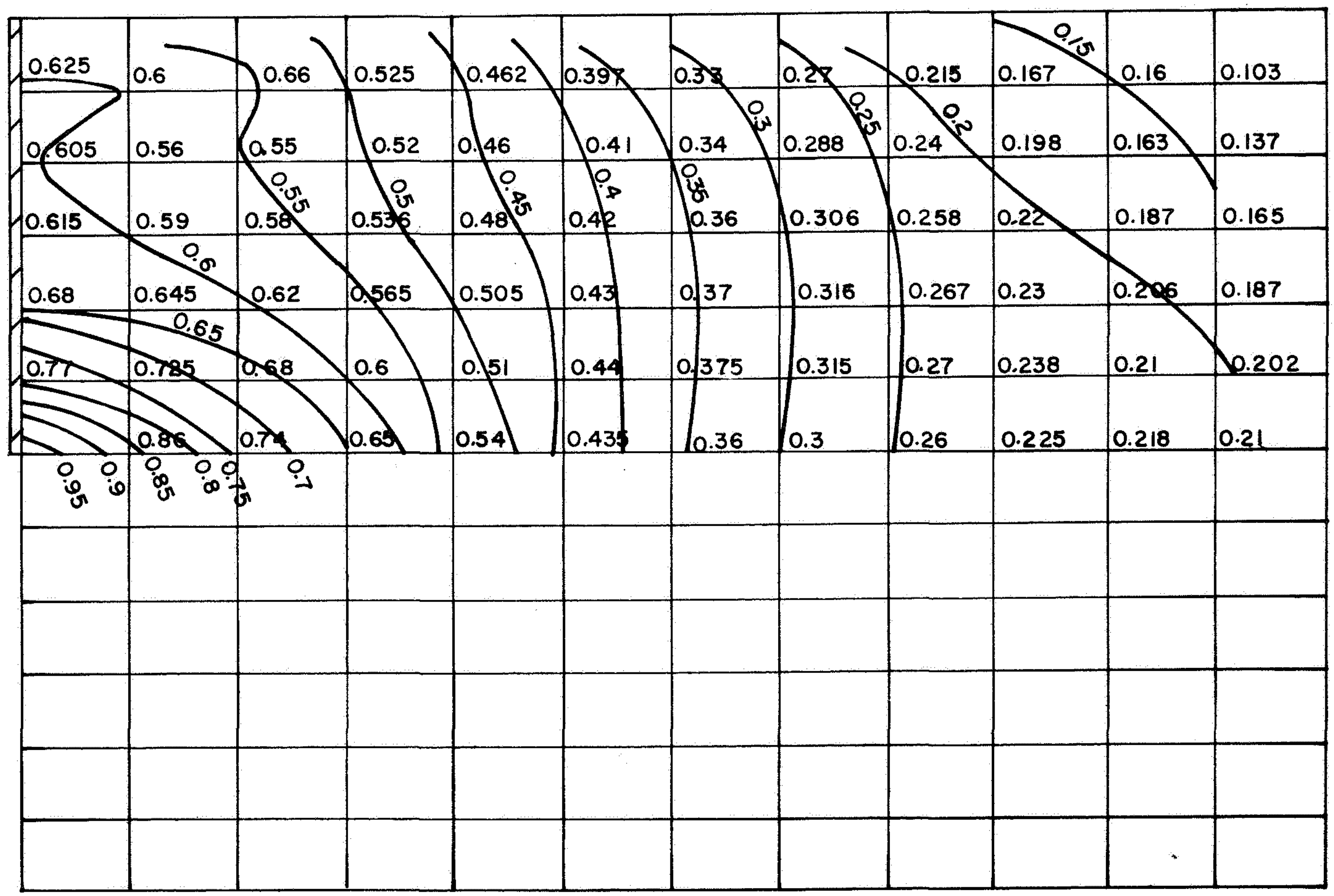


FIG. 5-26

CONTOURS OF HORIZONTAL STRESSES/MAXIMUM STRESS FOR THE REGION BOUNDED BY THE PLATE. RATE OF LOADING = 266 IN./SEC DISPLACEMENT OF PLATE = 0.1064 IN.

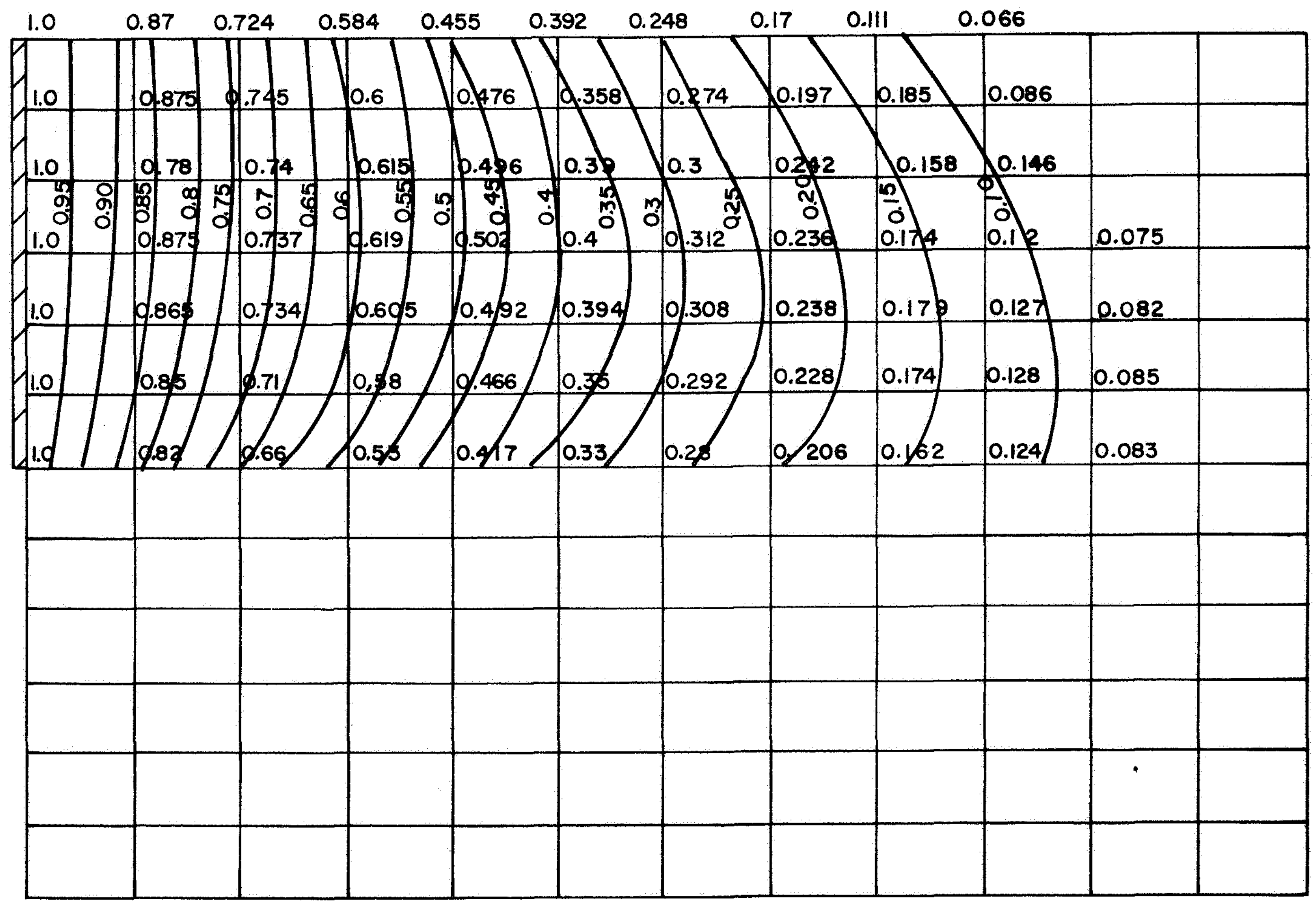


FIG. 5-27

CONTOURS OF HORIZONTAL DISPLACEMENTS/PLATE DISPLACEMENT FOR THE REGION BOUNDED BY THE PLATE. RATE OF LOADING = 266 IN./SEC MOVEMENT OF PLATE = 0.1064 IN.

2. For rates of loadings from 2.66 up to 106.4 in./sec, there is a marked effect on the state of stress and deformation in a region within a distance approximately equal to the length of the bearing surface. Higher rates are associated with higher stresses.

3. With increased velocity beyond 106.4 in./sec, Conclusion 1 is drawn again. Also, a plateau of force level develops with increasing velocity.

Restating Conclusions 1, 2 and 3 it can be said that no significant change is observed if slow rates ($\leq$ 2.66 in./sec) alone are compared. Significant effect is observed if slow rates are compared with intermediate ($>$ 2.66 and $\leq$ 106.4 in./sec) rates, and with high ($>$ 106.4 in./sec) rates. The significant effect is observed when intermediate rates are compared alone or with the high rates. No significant effect is observed if high rates are compared alone.

4. The distribution of stresses on the plate increasingly deviates from linearity with increased rates, if slow, intermediate and high rates are compared. For all rates of loading the highest stresses develop at the lower boundary of the plate.

5. In the region outside that bounded by a distance equal to approximately the length of the bearing surface, the state of stress and deformation changes slightly with the rate of loading.

6. The significant effect of the rate of loading on the force displacement history described in Conclusions 1, 2 and 3, increases with increase in deformation.

7. For slow rates, the earth pressure modulus decreases with increased displacements of the plate. For intermediate and high rates there is some displacement at which the modulus starts to increase until it starts to

decrease again at a higher displacement; both values of displacements are different for intermediate and high rates.

8. If slow, intermediate and high rates of loading are compared, there is an increase in the value of displacement at which the resistance starts to decrease with increased rates of loading.

The above conclusions were drawn from the numerical experiments. It should be emphasized that all numerical values obtained depend on the E' and ν' relation with respect to the axial strain ϵ_x which was obtained from two experimental curves. Therefore all quantitative information regarding the above conclusions depend on the E' and ν' relation with axial strains.

Experimental research has been done to obtain E' vs ϵ_x curves but little has been done to investigate the ν' vs ϵ_x relations for different soils. All this leads to the important conclusion that

9. Extensive research is needed to investigate the variation of ν' and E' with the strain level. Such information is of high importance in any theoretical solution.

Based on the above conclusions, the following recommendations are proposed:

1. Numerical experiments using the computer program should be done on different sizes of contact surfaces and different soil types in order that more general conclusions can be drawn regarding soil response to dynamic loading.

2. An extension to the program is recommended to solve for inclined dynamic loading; such extension would involve the resolution of body force into directions parallel and normal to the direction of impact. The coordinate axis would be rotated so as to coincide with the above directions.

3. An extension of the program is recommended to vary the deformation properties inside the relaxation net.

4. An extension of the program is needed to solve for sloping or irregular boundaries using numerical techniques.

5. Further studies are recommended to force agreement between analytical and experimental results by varying arbitrarily the lateral strain ratio curve.

6. A rotationally symmetric solution is recommended for further studies.

7. Further study is recommended to determine the influence of rigid inclusions inside the relaxation net.

8. A subroutine should be added to the computer program for the purpose of plotting contour lines of stress and displacement in the soil mass.

APPENDIX A

DERIVATION OF STRAIN COMPONENTS

Two modes can be employed to describe the deformation in a continuous medium: the Lagrangian and the Eulerian modes. In the Lagrangian description, the coordinates a_i of a typical particle in the initial state are treated as independent variables, while in the Eulerian description, the coordinates x_i of the particle in the deformed state are treated as the independent variables. Hence, if the Lagrangian mode is used, the coordinates in the deformed states are written in terms of those of the initial state as

$$x_i = x_i \ (a_1, a_2, a_3). \tag{A-1}$$

On the other hand, if the Eulerian mode is used, deformations are described by

$$a_i = a_i \ (x_1, x_2, x_3 \tag{A-2}$$

In any problem, stresses acting through the medium must satisfy equilibrium conditions in the deformed state. In this study Eulerian coordinates have been used to describe the strain components. For infinitesimal deformations (when products of derivatives can be neglected), the two modes are identical.

Consider a group of particles on some Curve C_0 before deformation, (cf. Fig. A-1).

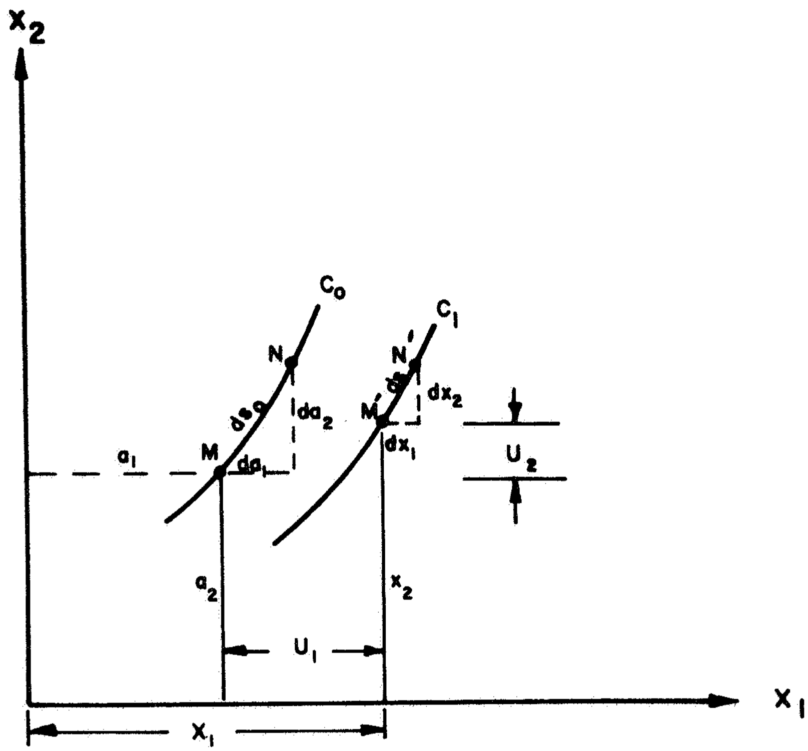


FIG. A-1

DEFORMATION OF AN ARBITRARY

LINE ELEMENT M N.

Let the coordinate of some particle M on C_0 be denoted by (a_1, a_2), and the coordinate of another particle N, at a distance ds_0 from point M be $(a_1 + da_1, a_2 + da_2)$. After deformation point M and N will be on another curve, say C_1. The new coordinates of point M which is now M′ on C_1 are (x_1, x_2) and the coordinates of N which is now N′ on C_1 are $(x_1 + dx_1, x_2 + dx_2)$.

The elements ds_0 and ds on C_0 and C_1 can be described as follows:

$$ds_0^2 = da_1^2 + da_2^2 = da_i\ da_i \quad i = 1, 2 \tag{A-3}$$

$$ds^2 = dx_1^2 + dx_2^2 = dx_i\ dx_i \quad i = 1, 2 \tag{A-4}$$

Considering the Eulerian description of deformation, then from Eq A -2

$$da_1 = \frac{\partial a_1}{\partial x_1} dx_1 + \frac{\partial a_1}{\partial x_2} dx_2 \tag{A-5}$$

$$da_2 = \frac{\partial a_2}{\partial x_1} dx_1 + \frac{\partial a_2}{\partial x_2} dx_2 \tag{A-6}$$

Or in a tensor notation

$$da_i = a_{ik} dx_k \tag{A-7}$$

where

$a_{ik} = \dfrac{\partial a_i}{\partial x_k}$ denote the differentiation with respect to the kth independent variable.

Substituting Eq A-7 in A-3 yields

$$\begin{aligned} ds_o^2 &= da_1^2 + da_2^2 \\ &= \left(\frac{\partial a_1}{\partial x_1}\right)^2 dx_1^2 + \left(\frac{\partial a_1}{\partial x_2}\right)^2 (dx_2)^2 \\ &\quad + 2 \frac{\partial a_1}{\partial x_1} \frac{\partial a_1}{\partial x_2} dx_1 \, dx_2 \\ &\quad + \left(\frac{\partial a_2}{\partial x_2}\right)^2 (dx_2)^2 + \left(\frac{\partial a_2}{\partial x_1}\right)^2 (dx_1)^2 \\ &\quad + 2 \frac{\partial a_2}{\partial x_2} \frac{\partial a_2}{\partial x_1} dx_1 \, dx_2 \end{aligned} \tag{A-8}$$

Or in tensor notation

$$ds_o^2 = a_{ij} \, a_{ik} \, dx_j \, dx_k \tag{A-9}$$

$i = 1, 2 \qquad j = 1, 2 \qquad k = 1, 2$

It is obvious the equality of ds_o^2 and ds^2 implies that the transformation $a_i = a_i \; (x_1, x_2)$ is one of a rigid body motion; hence it is logical to take the quantity $ds^2 - ds_o^2$ as a measure of strain, and therefore it can be written that

$$\begin{aligned} ds^2 - ds_o^2 = dx_1^2 + dx_2^2 &- \left(\frac{\partial a_1}{\partial x_1}\right)^2 dx_1^2 - \left(\frac{\partial a_2}{\partial x_2}\right)^2 dx_2^2 \\ &- \left(\frac{\partial a_1}{\partial x_2}\right)^2 (dx_2)^2 - \left(\frac{\partial a_2}{\partial x_1}\right)^2 (dx_1)^2 \\ &- 2\,\frac{\partial a_1}{\partial x_1}\frac{\partial a_1}{\partial x_2}\, dx_1\, dx_2 \\ &- 2\,\frac{\partial a_2}{\partial x_2}\frac{\partial a_2}{\partial x_1}\, dx_1\, dx_2 \end{aligned} \tag{A-10}$$

or in a tensor notation,

$$\begin{aligned} ds^2 - ds_o^2 &= dx_i\, dx_i - a_{ij}\, a_{ik}\, dx_j\, dx_k \\ &= (\delta_{jk} - a_{ij}\, a_{ik})\,(dx_j\, dx_k) \end{aligned}$$

$$\begin{array}{lll} i = 1,\ 2, & j = 1,\ 2, & k = 1,\ 2 \\ \delta_{jk} = 1, & i = j & \\ \delta_{jk} = 0, & i \neq j & \end{array}$$

$$= 2\,\omega_{jk}\, dx_j\, dx_k \tag{A-11}$$

where ω_{jk} is the strain function and

$$2\,\omega_{jk} = \delta_{jk} - a_{ij}\, a_{ik} \tag{A-12}$$

The strains ω_{jk} can be written in terms of displacements since

$$U_i = x_i - a_i \quad \text{and therefore}$$

$$a_i = x_i - U_i \qquad i = 1,\ 2 \tag{A-13}$$

where U_i stands for displacements.

Substituting Eq A-13 in Eq A-12, it can be written that

$$2\ \omega_{jk} = U_{jk} + U_{kj} - U_{ij}\ U_{ik}$$

or

$$2\ \omega_{jk} = \frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j} - \frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_k} \tag{A-14}$$

Now the expressions for strain in unabridged notations can be written as

$$\epsilon_x = \frac{\partial u}{\partial x} - 1/2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\epsilon_y = \frac{\partial v}{\partial y} - 1/2\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$2\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) = \epsilon_{xy}$$

If, on the other hand, the Lagrangian coordinates were used, so that a_i are treated as independent variables, then following the same steps it is easy to establish the following relations.

$$dx_i = x_{ij}\ da_j \quad i = 1,\ 2 \qquad j = 1,\ 2 \tag{A-15}$$

where x_{ij} is the derivative of x_i with respect to the jth independent variable, which is a_j in Eq A-15.

$$ds^2 = dx_i \; dx_i = x_{ij} \; x_{ik} \; da_j \; da_k$$

$$ds_o{}^2 = da_j \; da_j = \delta_{jk} \; da_j \; da_k$$

$$i = 1,\ 2 \qquad j = 1,\ 2 \qquad k = 1,\ 2$$

$$\delta_{jk} = 0 \qquad j \neq k$$

$$\delta_{jk} = 1 \qquad j = k$$

$$ds^2 - ds_o{}^2 = 2 \; \eta_{ij} \; da_j \; da_k \tag{A-16}$$

where

$$2 \; \eta_{ij} = (x_{ij} \; x_{ik} - \delta_{jk}) \; da_j \; da_k$$

Since

$$x_i = a_i + U_i$$

then

$$x_{ij} \; x_{ik} = (\delta_{ij} + U_{ij}) \; (\delta_{ik} + U_{ik})$$

$$= \delta_{jk} + U_{jk} + U_{kj} + U_{ij} \; U_{ik}$$

$$ds^2 - ds_o{}^2 = (U_{jk} + U_{kj} + U_{ij} \; U_{ik}) \; da_j \; da_k \tag{A-17}$$

and therefore

$$2 \; \eta_{jk} = (U_{jk} + U_{kj} + U_{ij} \; U_{ik}) \tag{A-18}$$

hence in unabridged form

$$\epsilon_x = \frac{\partial u}{\partial a_1} + 1/2 \left[\left(\frac{\partial u}{\partial a_1} \right)^2 + \left(\frac{\partial v}{\partial a_1} \right)^2 \right]$$

$$\epsilon_y = \frac{\partial v}{\partial a_2} + 1/2 \left[\left(\frac{\partial v}{\partial a_2} \right)^2 + \left(\frac{\partial u}{\partial a_2} \right)^2 \right]$$

$$2\ \gamma_{xy} = \left(\frac{\partial u}{\partial a_2} + \frac{\partial v}{\partial a_1} \right) - \left(\frac{\partial u}{\partial a_1} \frac{\partial u}{\partial a_2} + \frac{\partial v}{\partial a_1} \frac{\partial v}{\partial a_2} \right)$$

Physical Meaning of Strain Components

Consider a line element with $ds_o = da_1$, $da_2 = 0$ (that is, the line element is parallel to the x axis before deformation) and define the relative deformation ϵ_1 as $\frac{ds - ds_o}{ds_o}$ then

$$ds = (1 + \epsilon_1)\ ds_o$$

From Eq A-16,

$$ds^2 - {ds_o}^2 = 2\ \eta_{jk}\ da_j\ da_k = 2\ \eta_{11}\ {da_1}^2 \qquad \text{(A-19)}$$

then

$$(1 + \epsilon_1)^2 = 1 + 2\ \eta_{11}$$

$$\epsilon_1 = \sqrt{1 + 2\ \eta_{11}} - 1 \qquad \text{(A-20)}$$

By the same reasoning, if the line element was parallel to the y axis before deformation it can be shown that

$$\epsilon_2 = \sqrt{1 + 2\ \eta_{22}} - 1$$

The quantities ϵ_1 and ϵ_2 are the relative deformations of the elements M-N and M-L (Fig. A-2) which in the deformed state are $M' - N'$ and $M' - L'$.

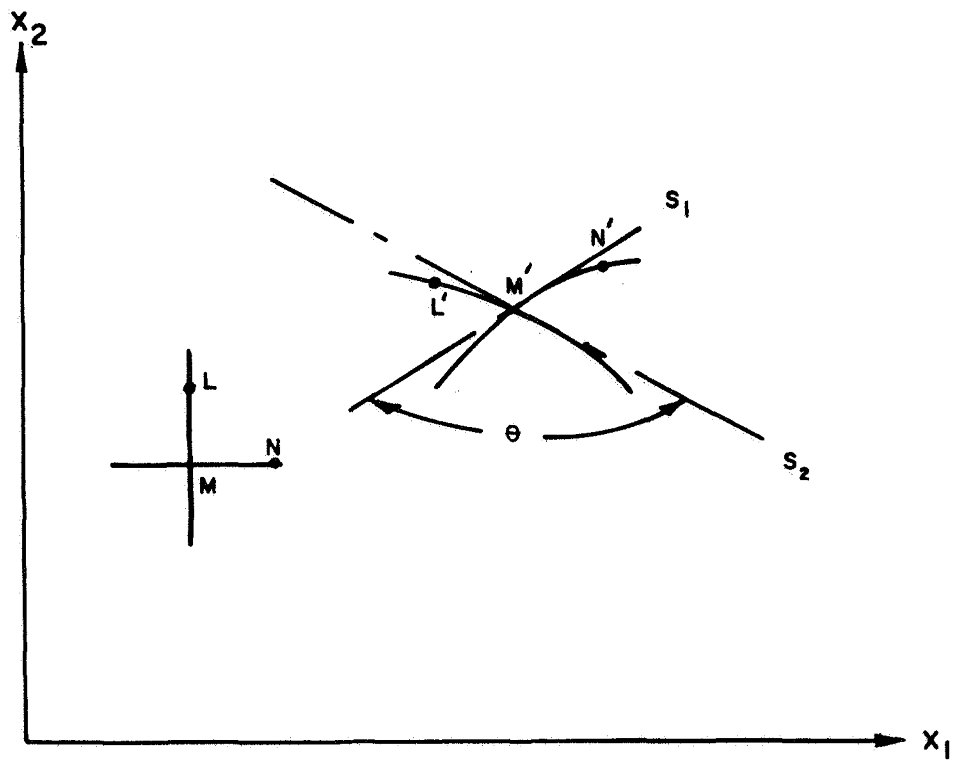


FIG. A-2

PHYSICAL MEANING OF STRAIN COMPONENTS

Prior to deformation the angle between $M - N$ and $M - L$ is a right angle, after deformation $M' - N'$ is an element of an arc as well as $M' - L'$. The angle between the new elements is no longer a right angle unless the motion is that of a rigid body motion. The strain components η_{11} and η_{22} indicate the relative deformation of these elements which were initially parallel to the coordinate axis.

The projection on the x and y axis of the element $M' - N'$ can be obtained as follows:

$$dx_1 = \frac{\partial x_1}{\partial a_1} da_1 = dx \qquad \text{(a)}$$

$$\text{(A-21)}$$

$$dx_2 = \frac{\partial x_2}{\partial a_1} da_1 = dy \qquad \text{(b)}$$

Equation A-21 can be written in terms of displacements since $x_i = a_i + U_i$, then

$$dx_1 = \left(1 + \frac{\partial U_1}{\partial a_1}\right) da_1 = dx \qquad (a)$$

$$dx_2 = \left(1 + \frac{\partial U_2}{\partial a_1}\right) da_1 = dy \qquad (b)$$

$$ds' = M'N' = \sqrt{dx_1^2 + dx_2^2} = \sqrt{(1+\epsilon_{xx})^2 + (\tfrac{1}{2}\,\epsilon_{xy})^2}\; da_1$$

$$= (1 + \epsilon_1)\; da_1$$

where

$$\frac{\partial U_1}{\partial a_1} = \epsilon_{xx}, \qquad \frac{\partial U_2}{\partial a_1} = \tfrac{1}{2}\,\epsilon_{xy}$$

$$\epsilon_1 = \frac{|M'N'| - |MN|}{|MN|}$$

The direction cosines for $M'N'$ and $M'L'$ (cf. Fig. A-2), can therefore be computed as:

$$\cos(M'N', x) = \frac{1 + \epsilon_{xy}}{1 + \epsilon_1}, \quad \cos(M'N', y) = \frac{\tfrac{1}{2}\,\epsilon_{xy}}{1 + \epsilon_1}$$

$$\cos(M'L', x) = \frac{\tfrac{1}{2}\,\epsilon_{yy}}{1 + \epsilon_2} \quad \cos(M'L', y) = \frac{1 + \epsilon_{yy}}{1 + \epsilon_2} \qquad \text{(A-22)}$$

Equation A-22 gives the direction cosines of the tangent to the arc at point M'.

The cosine of the angle between the two tangents can be obtained from analytic geometry as:

$$\cos\ (S_1,\ S_2) = \cos\ (S_1,\ x)\ \cos\ (S_2,\ x)$$

$$+ \cos\ (S_1,\ y)\ \cos\ (S_2,\ y)$$

when S_1 and S_2 are the tangents to the point M' along $M'\ N'$ and $M'\ L'$.
or

$$\cos\ \theta = \frac{\epsilon_{xv}}{(1 + \epsilon_1)(1 + \epsilon_2)}$$

Prior to deformation, the angle θ was a right angle, denoting $\Delta\theta xy$ as the change due to deformation, then,

$$\cos\ (\pi/2 - \Delta\theta) = \mathrm{Sin}\ \Delta\theta xy = \frac{\epsilon_{xy}}{(1 + \epsilon_1)(1 + \epsilon_2)} \qquad \text{(A-23)}$$

It is obvious from Eq A-23 that the strain component ϵ_{xy} indicates the shear, and if such strain component vanish; the angle between the two elements would remain a right angle.

APPENDIX B

EQUILIBRIUM CONDITIONS IN THE LINEAR CASE

For elastically linear material, the definitions of strains and stresses are:

$$
\begin{aligned}
&\epsilon_x = u_x \\
&\epsilon_y = v_y \\
&\epsilon_{xy} = u_y + v_x \\
&\sigma_x = \lambda\,(\epsilon_x + \epsilon_y) + 2G\,\epsilon_x \\
&\sigma_y = \lambda\,(\epsilon_x + \epsilon_y) + 2G\,\epsilon_y \\
&\sigma_z = \lambda\,(\epsilon_x + \epsilon_y) \\
&\sigma_{xy} = G\,\epsilon_{xy} \\
&\frac{\partial \sigma_x}{\partial x} = (\lambda + 2G)\,u_{xx} + \lambda\,v_{xy} \\
&\frac{\partial \sigma_y}{\partial y} = (\lambda + 2G)\,v_{yy} + \lambda\,u_{xy} \\
&\frac{\partial \sigma_{xy}}{\partial y} = G\,(u_{yy} + v_{xy}) \\
&\frac{\partial \sigma_{xy}}{\partial x} = G\,(v_{xx} + u_{xy})
\end{aligned}
\qquad \text{(B-1)}
$$

Therefore the equilibrium equations for a linear case in terms of displacement can be written as:

$$(\lambda + 2G)\ u_{xx} + (\lambda + G)\ v_{xy} + G\ u_{yy} + F_x - \rho\ \ddot{u} = REX_{i,j,k} \tag{B-2a}$$

$$(\lambda + 2G)\ v_{yy} + (\lambda + 2G)\ u_{xy} + G\ v_{xx} + F_y - \rho\ \ddot{v} = REY_{i,j,k} \tag{B-2b}$$

or in finite difference form,

$$(\lambda + 2G)\ (A) + (\lambda + G)\ (E) + (G)\ (M) + F_x - \rho\ (PH) = REX_{i,j,k} \tag{B-3a}$$

$$(\lambda + 2G)\ (N) + (\lambda + G)\ (L) + G\ (D) + F_y - \rho\ (PS) = REY_{i,j,k} \tag{B-3b}$$

Equation B-3 can be obtained directly from Eq 4-1 if the products of strains are set to zero. When such products are set to zero then;

$$\begin{aligned} A\ B &= u_{xx}\ u_x = 0 \\ C\ D &= v_x\ v_{xx} = 0 \\ Q\ F &= v_{xy}\ v_y = 0 \\ R\ L &= u_y\ u_{xy} = 0 \\ M\ B &= u_{yy}\ u_x = 0 \\ N\ C &= v_{yy}\ v_x = 0 \end{aligned} \tag{B-4}$$

If all the above terms are set to zero, then Eq B-3 will be identical to Eq 4-1.

Convergence

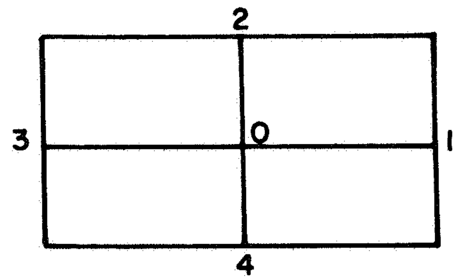


FIG. B-1

NODAL POINTS IN RESIDUAL DISTRIBUTION

Considering an arbitrary node 0, together with the adjacent nodes 1, 2, 3 and 4, (cf. Fig. B-1) the initial X residuals (REX) and Y residuals (REY) are assumed to be the same for all nodes.

Starting with point 0, an increment Δu is applied

$$\Delta u = REX_o \ / \ W_o$$

where

$$W_o = -2\ (VDR/HX^2 + SHM/HY^2) - \frac{RHO}{HT^2}$$

$$\Delta REX_o = \frac{-REX_o}{W_o} W_o = -REX_o \quad .$$

Then the new residual at node 0 will be

$$REX_o' = REX_o - REX_o = 0$$

The incremental deformations applied at node 0 is $\frac{-REX_o}{W_o}$ and this will change the residual at other adjacent nodes, then the new residual at node 1 will be $REX_1 = REX_0 \left(1 - \frac{W_1}{W_0}\right)$ where

$$W_1 = VDR\ (1/HX^2 - 1/2\ HX^3)$$

and the new residual at node 3 will be

$$REX_3 = REX_0 \left(1 - \frac{W_2}{W_0}\right)$$

where

$$W_2 = VDR \left(\frac{1}{HX^2} + \frac{1}{2HX^3}\right)$$

and similarly

$$REX_2 = REX_0 \left(1 - \frac{W_3}{W_0}\right)$$

$$REX_4 = REX_0 \left(1 - \frac{W_3}{W_0}\right)$$

where

$$W_3 = SHM/HY^2$$

At this stage the largest residual will be at node 3, since $W_2 > W_3$ and W_0 is negative.

Liquidating the residual REX_3, the new residual at node 0 will be

$$0 + \frac{REX_3}{W_0} W_1$$

$$REX_0'' = REX_0 \left(1 - \frac{W_2}{W_0}\right) \frac{W_1}{W_0}$$

Since $W_2 < |W_0|$ then;

$$\left| \left(1 - \frac{W_2}{W_o} \right) \right| < 2$$

$$\left| \frac{W_1}{W_o} \right| = \frac{\dfrac{VDR}{HX^2} - \dfrac{VDR}{2HX^3}}{2\left(\dfrac{VDR}{HX^2} + \dfrac{SHM}{HY^2} \right) + \dfrac{RHO}{HT^2}}$$

Since VDR and SHM and RHO are positive numbers, then

$$\left(\frac{VDR}{HX^2} - \frac{VDR}{2HX^3} \right) < \frac{VDR}{HX^2}$$

and hence it follows that,

$$\frac{\left(\dfrac{VDR}{HX^2} - \dfrac{VDR}{2HX^3} \right)}{\left(\dfrac{VDR}{HX^2} + \dfrac{SHM}{HY^2} \right) + \dfrac{RHO}{HT^2}} < 1$$

and hence $\left| \frac{W_1}{W_o} \right| < \frac{1}{2}$

since $\left| 1 - \frac{W_2}{W_o} \right| < 2$

then $\left| \frac{W_1}{W_o} \left(1 - \frac{W_2}{W_o} \right) \right| < 1$

or $S < 1$

where $S = \left| \frac{W_1}{W_o} \left(1 - \frac{W_2}{W_o} \right) \right|$

and therefore

$$|\ REX_o'' \ | < REX_o$$

After n iterations;

$$REX_{o_n} = REX_o \ S^n$$

Since S is less than 1, then

$$REX_{o_n}' \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

The same argument can be built for any node in the Region R.

Similarly, in the liquidation of the Y residuals;

$$\Delta v = - REY_o \ / \ W_o'$$

where

$$W_o' = - 2 \ (VDR/HY^2 + SHM/HX^2) - \frac{RHO}{HT^2}$$

$$\Delta REY_o = - REY_o$$

$$REY_o' = 0$$

The new residual REY_1 at Node 1 will be

$$REY_1 = REY_o \left(1 - \frac{W_1'}{W_o} \right)$$

where

$$W_1' = SHM/HX^2$$

and

$$REY_3 = REY_o \ (1 - W_1'/W_o)$$

$$REY_2 = REY_0 \ (1 - W_3{}'/W_0)$$

where

$$W_3{}' = VDR \ (1/HY^2 + 1/2HY^3)$$

$$REY_4 = REY_0 \left(1 - \frac{W_2{}'}{W_0} \right)$$

where

$$W_2{}' = VDR \ (1/HY^2 - 1/2 \ HY^3).$$

Since the largest residual at this stage is at node 2, then liquidating node 2 we obtain the new residual at node 0 as,

$$REY_0'' = \frac{REY_2}{W_0} \ (W_2{}')$$

$$= REY_0 \left(1 - \frac{W_3{}'}{W_0} \right) \frac{W_2{}'}{W_0}$$

It can be shown easily that;

$$\left| \ 1 - \frac{W_3{}'}{W_0} \ \right| < 2$$

$$\frac{W_2{}'}{W_0} < \frac{1}{2}$$

therefore

$$S' = \left| \left(1 - \frac{W_3{}'}{W_0} \right) \frac{W_2{}'}{W_0} \right| < 1$$

and

$$REY_0'' < REY_0$$

After n iterations,

$$REY_0 = REY_0 \, (S)^n$$

and therefore

$$REY_{0_n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

APPENDIX C

A PROPOSED ELASTO-PLASTIC ANALYSIS

The purpose of this section is to develop a stress-strain relation for soils after yield. Since, for a particular region, yielding does not occur simultaneously, a yielding criteria should be considered. The one considered here is the von Mises - Hencky criterion (5)* which states that yielding will occur when the principal stresses σ_1, σ_2 and σ_3 attain values such that,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_2 - \sigma_3)^2 = 8\ K^2 \qquad \text{(C-1)}$$

where K is a constant for a particular material.

Evidently K depends on the confining pressure for soils and hence it is a function of depth. For a plane strain problem σ_3 is not zero and hence we assume the lateral strain ratio to be 0.5 so that,

$$\sigma_3 = \frac{\sigma_1 + \sigma_2}{2}$$

So that Eq C-1 takes the form

$$(\sigma_1 - \sigma_2)^2 = \frac{16\ K^2}{3}$$

or

$$(\sigma_x - \sigma_y)^2 + 4\ \sigma_{xy}^{\ 2} = \frac{16\ K^2}{3}$$

Here the yield function ϕ is defined as

$$\phi = 1/4\ (\sigma_x - \sigma_y)^2 + \sigma_{xy}^{\ 2} - \frac{4\ K^2}{3} = 0 \qquad \text{(C-2)}$$

* In Ref. 5, Hill replaces $8\ K^2$ in Eq C-1 by $6\ K^2$.

The yield condition represents a surface which is the yield Locus. It can be shown that such yield surface is Convex (6).

Considering the increment $d\phi$, then

$$d\phi = \frac{\partial\phi}{\partial\sigma_x} d\sigma_x + \frac{\partial\phi}{\partial\sigma_y} d\sigma_y + \frac{\partial\phi}{\partial\sigma_{xy}} d\sigma_{xy} = 0$$

the vectors $\frac{\partial\phi}{\partial\sigma_x}$ and $d\sigma_x$ are orthogonal.

Since there is no strain hardening; then

$$d\epsilon_x \, d\sigma_x = 0$$

$$d\epsilon_y \, d\sigma_y = 0$$

$$d\epsilon_{xy} \, d\sigma_{xy} = 0$$

And in terms of the plastic potential ϕ,

$$d\epsilon_{ij} = \psi \frac{\partial\phi}{\partial\sigma_{i,j}}$$

where

$$\phi = \phi\,(\sigma_{i,j}).$$

The term $d\epsilon_{ij}$ can be replaced with $\frac{d\epsilon_{ij}}{dt}$ or $\dot{\epsilon}_{ij}$ then

$$\dot{\epsilon}_{ij} = \psi \frac{\partial\phi}{\partial\sigma_{i,j}} .$$

Applying the above principles, the following is obtained

$$\dot{\epsilon}_x = \psi \frac{\partial \phi}{\partial \sigma_x} = \frac{1}{4} \psi \ (2\sigma_x - \sigma_y) \qquad \text{(a)}$$

(C-3)

$$\dot{\epsilon}_y = \frac{1}{4} \psi \ (2\sigma_y - \sigma_x) \qquad \text{(b)}$$

$$\dot{\epsilon}_{xy} = \psi \ (\sigma_{xy}) \qquad \text{(c)}$$

or

$$\sigma_x = \frac{4}{3\psi} \ (2 \ \dot{\epsilon}_x - \dot{\epsilon}_y) \qquad \text{(a)}$$

$$\sigma_y = \frac{4}{3\psi} \ (2 \ \dot{\epsilon}_y - \dot{\epsilon}_x) \qquad \text{(b)} \qquad \text{(C-4)}$$

$$\sigma_{xy} = \frac{1}{\psi} \ (\dot{\epsilon}_{xy}) \qquad \text{(c)}$$

where ψ is a function of the strain rate.

To find ψ, substitute the values of σ_x, σ_y and σ_{xy} in terms of strain rates in Eq C-2 and ψ can be obtained as

$$\psi = \frac{1}{\sqrt{3}\ K} \left[(\dot{\epsilon}_x - \dot{\epsilon}_y)^2 + 9/4 \ (\dot{\epsilon}_{xy})^2 \right]^{\frac{1}{2}}$$

ψ can be treated as a constant which should be computed for the instantaneous strain rates, in the same sense that the modulus of deformation E' and lateral strain ratio ν' are computed for each particular axial strain.

Equation C-4 satisfies the condition $\phi = 0$ and also has to satisfy the conditions of equilibrium. The definitions for strains used in Chapter II are for finite deformations and hence can be used for the conditions after yield and hence the equilibrium equations in terms of displacements can be developed as follows:

$$\dot{\epsilon}_x = \dot{u}_x (1 - u_x) - v_x \dot{v}_x \qquad \text{(a)}$$

$$\dot{\epsilon}_y = \dot{v}_y (1 - v_y) - u_y \dot{u}_y \qquad \text{(b)} \qquad \text{(C-5)}$$

$$\dot{\epsilon}_{xy} = \dot{u}_y (1 - u_x) + \dot{v}_x (1 - v_y) - u_y \dot{u}_x - v_x \dot{v}_y \qquad \text{(c)}$$

In finite difference form Eq C-5 can be written as:

$$\dot{\epsilon}_x = (1 - B_{i,j,k}) \left(\frac{B_{i,j,k} - B_{i,j,k-1}}{HT} \right) - C_{i,j,k}$$

$$\left[C_{i,j,k} - C_{i,j,k-1} \right] \frac{1}{HT} \qquad \text{(a)}$$

$$\dot{\epsilon}_y = (1 - F_{i,j,k}) \left(\frac{F_{i,j,k} - F_{i,j,k-1}}{HT} \right) - R_{i,j,k}$$

$$\left[\frac{R_{i,j,k} - R_{i,j,k-1}}{HT} \right] \qquad \text{(b)} \qquad \text{(C-6)}$$

$$\dot{\epsilon}_{xy} = (1 - B_{i,j,k}) \left(\frac{R_{i,j,k} - R_{i,j,k-1}}{HT} \right) + (1 - F_{i,j,k})$$

$$\left(\frac{C_{i,j,k} - C_{i,j,k-1}}{HT} \right) - R_{i,j,k} \left[\frac{B_{i,j,k} - B_{i,j,k-1}}{HT} \right]$$

$$- C_{i,j,k} \left[\frac{F_{i,j,k} - F_{i,j,k-1}}{HT} \right] \qquad \text{(c)}$$

and also

$$\sigma_{xx,x} = \frac{4}{3\psi} (2 \dot{\epsilon}_{x,x} - \dot{\epsilon}_{y,x}) \qquad \text{(a)}$$

$$\text{(C-7)}$$

$$\sigma_{yy,y} = \frac{4}{3\psi} (2 \dot{\epsilon}_{y,y} - \dot{\epsilon}_{x,y}) \qquad \text{(b)}$$

$$\sigma_{xy,x} = \frac{1}{\psi} (\dot{\varepsilon}_{xy,x}) \qquad (c)$$

(C-7)

$$\sigma_{xy,y} = \frac{1}{\psi} (\dot{\varepsilon}_{xy,y}) \qquad (d)$$

Substituting Eq C-7 in Eq C-6, the equilibrium equations after yield are obtained in terms of displacements.

$$\frac{1}{\psi} \Big[\mathrm{BB}\ (2.7\ \mathrm{AA} + \mathrm{MM}) - \mathrm{FF}\ (0.33\ \mathrm{EE}) - \mathrm{BK\ AM} - \mathrm{CC\ DN}$$

$$- C_{i,j,k}\ (2.7\ \mathrm{DD} + \mathrm{FN}) + 0.33\ (Q_{i,j,k}\ \mathrm{FK} + L_{i,j,k}\ \mathrm{GG}$$

$$+ R_{i,j,k}\ \mathrm{LL}) \Big] - (\mathrm{PH})\ \mathrm{RHO} + F_x = \mathrm{REX}_{i,j,k} \qquad (a)$$

(C-8)

$$\frac{1}{\psi} \Big[\mathrm{FF}\ (2.7\ \mathrm{FN} + \mathrm{DD}) - \mathrm{BB}\ (0.33\ \mathrm{LL}) - \mathrm{FK}\ (\mathrm{DNN})$$

$$- \mathrm{GG}\ (\mathrm{AAM}) - R_{i,j,k}\ (2.7\ \mathrm{MM} + \mathrm{AA}) + 0.33\ (L_{i,j,k}\ \mathrm{BK}$$

$$+ Q_{i,j,k}\ \mathrm{CC} + C_{i,j,k}\ \mathrm{EE} \Big] - (\mathrm{PS})\ \mathrm{RHO} + F_y = \mathrm{REY}_{i,j,k} \qquad (b)$$

where

$$\mathrm{BB} = 1 - B_{i,j,k}$$

$$\mathrm{AA} = (A_{i,j,k} - A_{i,j,k-1}) \,/\, \mathrm{HT}$$

$$\mathrm{MM} = (M_{i,j,k} - M_{i,j,k-1}) \,/\mathrm{HT}$$

$$\mathrm{FF} = 1 - F_{i,j,k}$$

$$\mathrm{EE} = (Q_{i,j,k} - Q_{i,j,k-1}) \,/\, \mathrm{HT}$$

$$\mathrm{FN} = (N_{i,j,k} - N_{i,j,k-1}) \,/\, \mathrm{HT}$$

$$\mathrm{BK} = (B_{i,j,k} - B_{i,j,k-1}) \,/\, \mathrm{HT}$$

$$CC = (C_{i,j,k} - C_{i,j,k-1}) \ / \ HT$$

$$DD = (D_{i,j,k} - D_{i,j,k-1}) \ / \ HT$$

$$FK = (F_{i,j,k} - F_{i,j,k-1}) \ / \ HT$$

$$GG = (R_{i,j,k} - R_{i,j,k-1}) \ / \ HT$$

$$LL = (L_{i,j,k} - L_{i,j,k-1}) \ / \ HT$$

$$AM = 2.7 \ A_{i,j,k} + M_{i,j,k}$$

$$DN = 2.7 \ D_{i,j,k} + N_{i,j,k}$$

$$DNN = 2.7 \ N_{i,j,k} + D_{i,j,k}$$

$$AAM = 2.7 \ M_{i,j,k} + A_{i,j,k}$$

All the other terms have been defined in Chapter IV.

For nodal points at one increment length from any boundary where stresses are specified, Eq C-8 has to be modified and then it takes the form:

At those nodal points where σ_y is specified;

$$\frac{\left[\left(\frac{4}{3\psi}\right)\left(2\,\dot{\epsilon}_{y_{i,j,k}} - \dot{\epsilon}_{x_{i,j,k}}\right) - \sigma_{y_{i,j-1,k}} \right]}{HY} + \frac{1}{\psi}$$

$$\Big[(BB)\ LL - GG\ A_{i,j,k} + FF\ DD - Q_{i,j,k}\ CC - R_{i,j,k}\ AA \tag{C-9a}$$

$$- L_{i,j,k}\ BK - D_{i,j,k}\ FK - C_{i,j,k}\ EE \Big] - (PS)\ RHO + PHO = REY_{i,j,k}$$

The other equation which has to be satisfied is the same as C-8a.

For those nodal points close to where σ_x is specified as zero;

$$\frac{\left[\left(\frac{4}{3\psi}\right)\left(2\,\dot{\epsilon}_{x_{i,j,k}} - \dot{\epsilon}_{y_{i,j,k}}\right)\right]}{HX} + \frac{1}{\psi}\Big[BB\ MM - N_{i,j,k}\ CC$$

$$+ FF\ EE - L_{i,j,k}\ GG - C_{i,j,k}\ FN - Q_{i,j,k}\ FK - M_{i,j,k}$$

$$BK - R_{i,j,k}\ LL\Big] - (PH)\ RHO = REX_{i,j,k}\ . \qquad \text{(C-9b)}$$

The other equation which has to be satisfied is the same as C-8b.

Following the same procedure as in the below yielding case, two residual liquidation patterns are obtained. For the liquidation of the X residuals (REX), and the liquidation of the Y residuals, the patterns are shown in Figs. C-1 and C-2.

Method of Solution

1. Compute the value of the yield function $\phi(\sigma_{i,j})$ at each nodal point. This can be done after selecting initial values of u and v at each nodal point.

2. If ϕ is negative, that means that the material at that nodal point has not yielded yet, and the solution goes on using the relaxation patterns in Fig. 4-2 and Fig. 4-3 together with the Eq. 4-1.

3. If ϕ is zero or positive that means that the material has yielded, the equilibrium Eq C-8 together with the relaxation pattern shown in Figs. C-1 and C-2 has to be used.

Liquidation process follows the same steps as outlined in Chapter IV. At each node the definitions of $REX_{i,j,k}$ and $REY_{i,j,k}$ together with the relaxation pattern are different depending on whether the value of ϕ is negative or positive.

If the below yielding analysis can be called nonlinear elastic analysis, then the above analysis is strictly nonlinear elastic-plastic analysis. A

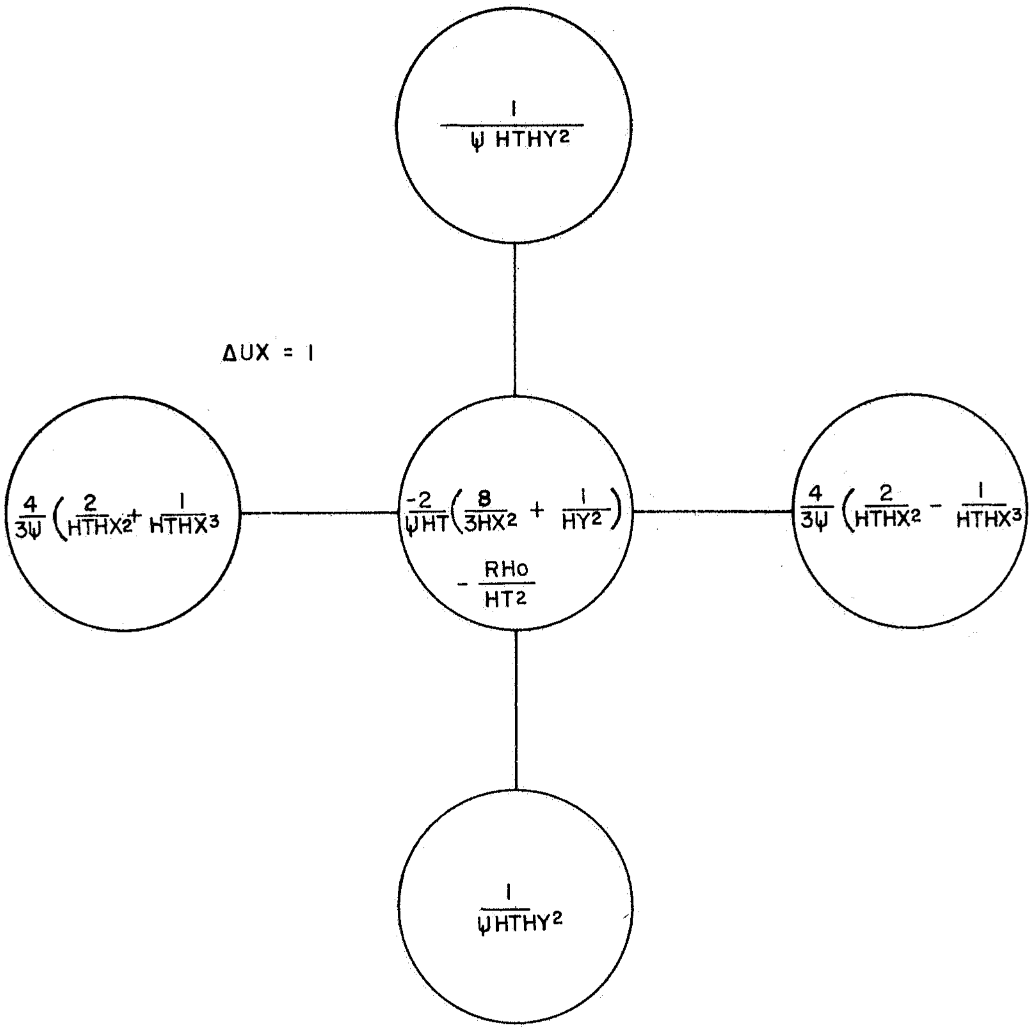


FIG. C-1

LIQUIDATION OF X RESIDUALS AFTER YIELD

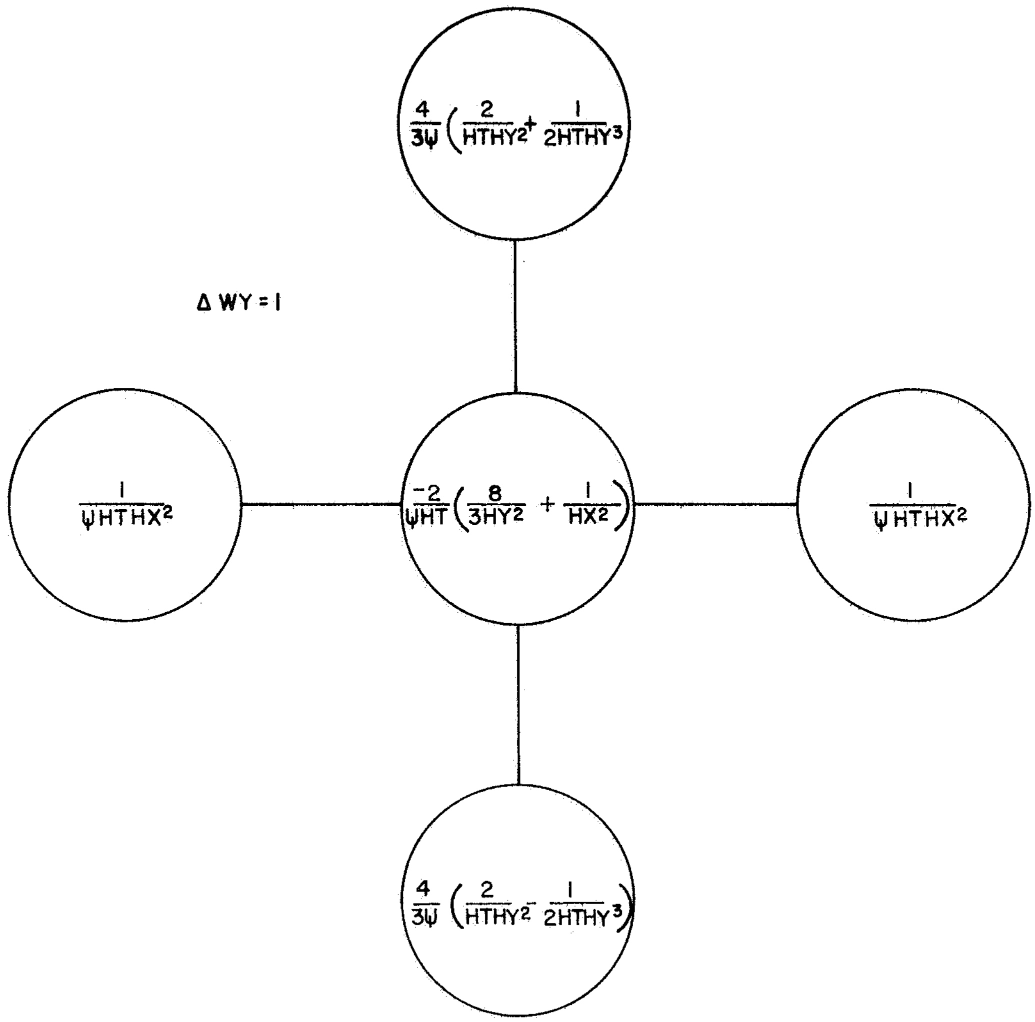


FIG. C-2

LIQUIDATION OF Y RESIDUALS AFTER YIELD

plastic-elastic surface can be obtained for a plane strain problem at a particular time station.

A computer program has been written for the solution of an elasto-plastic problem. It is essentially an extension for the first program for the below yielding case. The storage requirement for the elasto-plastic problem is a problem by itself. A computer with large storage capacity is needed to solve any practical problem. Extensive experimental work is needed to determine the value of K to be used in the yield function $\phi(\sigma_{i,j})$.

Due to the above difficulties the writer has not been able to obtain an elasto-plastic solution for any of the problems mentioned earlier. It is, however, the writer's belief that a solution can be obtained if these difficulties were tackled, especially if an appropriate value of K is found.

APPENDIX D

EVALUATION OF $P_{ult}/P_{ult_{max}}$ VS. RATE OF LOADING

```
      PROGRAM ERF ( INPUT, OUTPUT)
    7 READ 1,R , C,D, RR
C     -----  -  RR   RATE OF LOADING CORRESPONDING TO MAX. LOAD
C     ------ ---R    RATE OF LOADING
C---------- - C  AND  D ARE LIMITS OF INTEGRATION
    1 FORMAT( 3 E10.3)
         IF( R)99,99,8
    8           Z = (R/RR)*3.0
         IF(  Z-C) 2,2,3
    2           ZZ= D * Z
         GO TO 4
    3           ZZ = Z - C
    4           N = 0
                PART = ZZ
                SUM  = ZZ
    5           N = N +1
                PART= - PART*ZZ*ZZ/N
                TERM=   PART/(2*N+1)
                SUM = SUM + TERM
         IF( ABSF(TERM) - 0.0000001) 6,5,5
    6           E=1.1283792*SUM
         IF( Z-C) 10,10,11
   10           E = E + 0.056
   11 PRINT 12, R,E
   12 FORMAT( 10X,16HRATE OF LOADING = E10.3 ,10X,13HPULT/PULTMAX=E10.3)
         GO TO 7
   99 CONTINUE
      END
```

APPENDIX E

LISTING, FLOW CHART, AND INPUT GUIDE

FOR PROGRAM NASA

```
NASA,1,200,250000,320.CE364437,OWEIS.
QXX(RUN,S)
QXX(NASA)
'
      PROGRAM NASA    ( INPUT , OUTPUT )
      REAL L
      REAL M
      REAL  NN
      DIMENSION       AN1 ( 32 )   ,  AN2 ( 14 )   ,
     1  UX( 13,13,50) , WY( 13,13,50 ) , REX(13,13,50),REY(13,13,50),
     2  STY( 13, 1,50) ,  EX( 13,13,50 )
C---------       SOLVES FOR DYNAMIC RESPONSE  IN A PLANE STRAIN PROBLEM
C---------  -    DUE TO SPECIFIED  DISPLACEM NTS ON THE BOUNDARY
C---------       MTEST    BLANK FIELD FOR ALPHA NUMERIC ZERO
C---------       AN1(N)   ALPHA NUMERIC IDENTIFICATION
C--------        NPROB    PROBLEM NUMBER  ZERO TO EXIT
C---------       AN2(N)   ALPHA NUMERIC IDENTIFICATION
C---------       AP1--AP7 COEFFICIENTS OF THE MODLUS OF DEFORMATION  VS
C---------       AXIAL STRAIN POLYNOMIAL
C---------       CP1--CP5 COEFFICIENTS OF  THE LATERAL STRAIN RATIO
C---------       VS AXIAL STRAIN POLYNOMIAL
C---------       KPO      ZERO FOR NO  DATA PRINTOUT
```

```
C---------     KASE     ZERO FOR NO PRINTOUT OF RESIDUALS AFTER EACH
C---------  -  ITERATION
C---------     KOLE     ZERO FOR NO PRINTOUT OF X OR Y RESIDUALS
C---------     KTEST    ZERO  IF DIRECTION OF BODY FORCE IS NORMAL
C---------  -  TO BEARING SURFACE
C---------- -+ JTEST    ZERO FOR NO RIGID INCLUSION
C---------     ITEST    ZERO FOR PLATE PENETRATION, OTHERWISE A
C---------- --          WEDGE OR CYLINDER
C---------     NTEST    ZERO FOR    UNIFORM DISPLACMENTS  ON THE
C---------- --          BEARING SURFACE
C---------  -  LTEST    ZERO FOR NO PRINTOUT OF FINAL RESIDUALS
C---------- -  IM       MAX. NUMBER OF ITERATIONS FOR THE LIQUIDATION
C---------- --          OF X OR Y RESIDUALS
C---------- -  JB       = 1  FOR THE SOLUTION OF PROBLEMS  1  AND  2
C---------  -           =2   FOR THE SOLUTION OF PROBLEMS  3  AND  4
C---------- -           =3   FOR THE SOLUTION OF PROBLEMS  5  AND  6
C---------     CB1--2   NON ZERO  FOR SURFACE  FREE OF STRESSES
C---------     TOL      SPECIFIED CONVERGENCE  TOLERENCE
C---------     TOLP     BELOW TOLP , ACCELERATIN IS CONCIDIRED ZERO
C---------     KB       =1 IF WY OF THE NODES ON THE PLATE ARE  AS REAL
C---------              =2  IF WY IS 15 PERCENT OF THE ADJACENT NODE
C---------              =3 IF WY IS 30 PERCENT OF THE ADJACENT NODE
C---------              =4 IF WY IS 45 PERCENT OF THE ADJACENT NODE
C---------              =5 IF WY IS 60 PERCENT OF THE ADJACENT NODE
C---------              =6 IF WY IS 75 PERCENT OF THE ADJACENT NODE
C---------              =7 IF WY IS 90 PERCENT OF THE ADJACENT NODE
C---------              =8 IF WY IS EQUAL TO THAT OF THE ADJACENT NODE
C---------     LB       NON ZERO  FOR THE PROBLEM TO BE TREATED AS
C---------- -           RETAINING WALL
C---------     MX       NUMBER OF INCREMENTS IN X DIRECTION
C---------  -  MY       NUMBER OF INCREMENTS IN Y DIRECTION
C---------     MT       NUMBER OF TIME INCREMENTS
C---------     ALX      DIMENSION OF THE PROBLEM IN X DIRECTION
C---------     ALY      DIMENSION OF THE PROBLEM IN Y DIRECTION
C---------     HT       MAGNITUDE OF THE TIME INCREMENT
C---------- -  RHO      MASS DENSITY
   C-----------PHO      UNIT WEIGHT
C---------- -  RATE     RATE OF LOADING
C---------- -  WD       WIDTH OF BEARING SURFACE
C---------     MMY      NUMBER OF INCREMENTS TO THE BOUNDARY FROM THE
C---------              EDGE OF THE BEARING SURFACE , ALONG A LINE
C---------              PARALLEL TO THAT SURFACE ,  USED ONLY FOR NON
C---------  -           ZERO ITEST IN PROBLEMS 3,4,5 ,AND 6
C---------     YM1      SAME AS MMY  FOR PROBLEMS 5 AND 6 FOR
C----------             ZERO ITEST
C---------     UX       DISPLACEMENTS          IN DIRECTION OF PENETRAT
C---------     WY       DISPLACEMENTS NORMAL TO   DIRECTION OF PENETRAT
C---------     ITT1--2  DEFINES THE DIMENSION OF A RIGID INCLUSION
C---------              IN X DIRECTION ,ITT1 IS THE STARTING STATION
C---------     JTT1--2  DEFINES THE DIMENSION OF A RIGID INCLUSION
C---------  -           IN Y DIRECTION ,JTT1 IS THE STARTING STATION
C---------  -  REX      RESIDUAL IN X DIRECTION
C---------     REY      RESIDUAL IN Y DIRECTION
    1 FORMAT(5X,48HPROGRAM NASA1-MASTER DECK -IS OWEIS, WR COX
   10 FORMAT (    5H      , 80X, 10HI-----TRIM                  )
```

```
  12 FORMAT ( 16A5 )
 240 FORMAT(  5X, 2 I5    )
 153 FORMAT( 5X, E10.3)
1277 FORMAT( 2 E10.3 )
 414 FORMAT ( A5, 5X ,14A5 )
  11 FORMAT ( 5HI     , 80X,10HI-----TRIM )
  13 FORMAT (     5X , 16A5 )
 515  FORMAT (///10H      PROB , /5X, A5, 5X, 14A5 )
  20 FORMAT ( 10X , 14I5  /  2 E10.3 )
  21 FORMAT (//30H    TABLE 1  CONTROL DATA        ,
    1        /44H    NUMBER OF ITERATIONS FOR X OR Y CLOSURE , I5,
    2        /44H               TOLERENCE   LB              ,E10.3,
    3        /52H    PRINT OUT OPTION FOR DATA (NO PRINT OUT IF ZERO,
    45 )
  24 FORMAT(5X,3I5,6E10.3 / 5X,E10.3)
  25 FORMAT (//28H         NUM INCREMENTS  MX   , 40X, I5,
    1          28H         NUM INCREMENTS  MY   , 40X, I5,
    2          28H         NUM INCREMENTS  MT   , 40X, I5,
    3          28H       INCREMENT LENGTH  HX   , 40X, E10.3,
    4          28H       INCREMENT LENGTH  HY   , 40X, E10.3,
    5          28H       INCREMENT LENGTH  HT   , 40X, E10.3,
    6          28H  MASS DENSITY                 , 40X, E10.3,
    7          28H   UNIT WT.(LB/CU.IN.)       , 40X, E10.3,
    8          28H      LOADING RATE(IN/SEC ) , 40X, E10.3,
    9          28H   LOADING WIDTH (IN)        , 40X, E10.3 )
  27 FORMAT( 8E10.3)
  41 FORMAT (///30X,23HSPECIFIED DISPLACEMENTS    ,
    1    / 55H I               J           K           X DISP   Y DISP
  40 FORMAT (10X,I2,20X,I2,20X,I2,20X,E10.3,20X,E10.3)
 501 FORMAT ( 5X, 3I2,   4 E 10.3 )
 500 FORMAT (///30X,18H INITIAL RESIDUALS
    1      //15X,38HI    J  K      REX           REY        )
9363 FORMAT (///30X,18H   FINAL RESIDUALS
    1      //15X,38HI    J  K      REX           REY        )
9565 FORMAT( 5X, 3I2,2X,E10.3,2X,E10.3)
 799 FORMAT(1X,3I2,   3E10.3)
 681 FORMAT (///30X,27HLIQUIDATION OF X RESIDUALS    ,
    1        40H I            J          K         RESX          )
  68 FORMAT ( 1X,I2 , 15X, I2   , 15X,   I2 , 15X ,  E10.3   , I2 ,E10.3)
  71  FORMAT (//15X,48HNO X CLOSURE WITHIN SPECIFIED INITIAL TOLERANCE
 901 FORMAT (///30X,27HLIQUIDATION OF Y RESIDUALS    ,
    1         40HI          J          K         RESY          )
  92  FORMAT(1X,I2,15X,I2,15X,I2,15X,E10.3,E10.3 )
 202 FORMAT (5X, I5)
 108 FORMAT (//15X,38HNO CLOSURE WITHIN SPECIFIED TOLERANCE  )
 207 FORMAT (//35X, 8HRESULTS  ,
    1    55HUX                 WY          RESX         RESY
     PRINT 1012
1212 FORMAT( 1H1  )
 107 FORMAT(//5X,I2,2X,I2,2X,I2,5X,E10.3,5X,E10.3,5X,E10.3,5X,E10.3)
6001  FORMAT(1X,3I2,4E10.3)
7447 FORMAT( 5X, 4I5 )
             MTEST = 5H
     PRINT 1212
1000 PRINT 10
```

```
   94 FORMAT (///42H   NO CONVERGENCE WITHEN SPECIFIED TOL        )
 8015 FORMAT( 10X,7E10.3 / 10X,5E10.3 ).
C     PROGRAM AND PROBLEM IDENTIFICATION
      READ 12, (AN1(N), N = 1, 32 )
 1010 READ 414,   NPROB, (AN2(N) , N = 1,14 )
                 SUMT = 0.0
         IF( NPROB - MTEST)  1020, 999, 1020
 1020 PRINT 11
      PRINT  1
      PRINT 13, ( AN1(N), N = 1,32 )
      PRINT 515,NPROB, ( AN2(N) , N = 1, 14 )
C     INPUT TABLE 1, CONTROL DATA
      READ  8015  , AP1,AP2,AP3,AP4,AP5,AP6,AP7,CP1,CP2,CP3,CP4,CP5
      READ 20,KPO,KASE,KOLE,KTEST,JTEST,ITEST,NTEST,LTEST,IM,JB,CB1,CB2
     1  KB   , LB ,   TOL  , TOLP
      PRINT 21, IM, TOL  , PO
C---------       INPUT AND PRINTOUT OF CONSTANTS
      READ24, MX,MY,MT, ALX,ALY,HT,RHO,PHO,RATE,WD
C     COMPUTATION FOR CONVENIENCE
                 HX= ALX/ MX  $MXP1 = MX+1
                 HY=  ALY/ MY
                 MTP3 = MT +3
      PRINT 25, MX,MY,MT,HX,HY,HT,RHO,PHO,RATE,WD
C---------       TYPE OF PROBLEM TO BE SOLVED
          GO TO ( 4, 5, 6 ), JB
    4        JSS = 1
             JSS1 = JSS + MY
             JS2 = JSS1 + MY
             JSS2 = JSS1 + 1
               JSI = JS2 - 1
               JTT= 2
         IF ( CB1) 104, 105, 104
  105        J = 1
          DO 17   I = 1, MXP1
          DO 17   K = 3, MTP3
      READ 177, STY(I,J,K)
  177 FORMAT ( 5X, E10.3)
   17 CONTINUE
          GO TO 34
  104        J = 1
         DO 18  I = 1, MXP1
         DO  18  K = 1 , MTP3
             STY(I,J,K) = 0.0
   18 CONTINUE
          GO TO 34
    5    IF(  ITEST) 151, 152, 151
  151 READ 153 , MMY
                 JSS= MMY+1
                 JSS1 =   JSS+ MY
                 JS2 = JSS1 + MMY
                 JSS2 = JSS1 +1
                 JS4 =  JSS-1
                 JSI = JS2 - 1
                 JTT= 2
         GO TO 34
```

```
 152          JSS= MY+1
            JSS1 = MY + MY+ 1
            JS2 = JSS1 + MY
            JSS2 = JSS1 + 1
              JSI = JS2 -2
              JTT = 2
        GO TO 34
   6    IF ( ITEST )  6666 , 6667, 6666
6666 READ 153, MMY
        JSS = MMY +1
        JSS1 = JSS+ MY
        JSS2 = JSS1 +1
        JS4 =  JSS- 1
              JS2 = JSS1 + MMY
        JSI =  JS2 - 1
        JTT = 2
        GO TO 6668
6667 READ 202, YM1
            JSS = YMI +1      $  JSS1 = YM1 + MY + 1
              JS2 = JSS1 + MY
              JSS2 = JSS1 +1
              JSI = JS2 -2
        JTT = 2
6668    IF( CB2 )  104, 105, 104
  34        I = 1
              K = 3
        IF( KPO)  5111, 5110, 5111
5111 PRINT 41
5110    IF( NTEST) 4110, 4112, 4110
4112          J= JSS
              JSS3 = JSS+1
     READ 27,UX(I,J,K),WY(I,J,K),UX(I,J,K+1),WY(I,J,K+1),UX(I,J,K+2),W
    1   (I,J,K+2),UX(I,J,K+3),WY(I,J,K+3)
              JSS3 = JSS+1
        DO 4116 J = JSS3, JSS1
        UX(I,J,K)= UX(I,J-1,K)
        WY(I,J,K)= WY(I,J-1,K)
              UX(I,J,K+1)= UX(I,J-1,K+1)
              WY(I,J,K+1)= WY(I,J-1,K+1)
              UX(I,J,K+2)= UX(I,J-1,K+2)
              WY(I,J,K+2)= WY(I,J-1,K+2)
              UX(I,J,K+3)= UX(I,J-1,K+3)
              WY(I,J,K+3)= WY(I,J-1,K+3)
4116 CONTINUE
              K= K+4
        IF( K-MTP3 ) 4112, 4112, 4111
4110    IF(  I TEST )  2114, 2113, 2114
2114    DO 2117  J = JSS, JSS1
     READ 1277 , UX( I,J,K)  , WY( I,J,K)
2117 CONTINUE
        GO TO 4119
2113    DO 30  J = JSS , JSS1
     READ 27,UX(I,J,K),WY(I,J,K),UX(I,J,K+1),WY(I,J,K+1),UX(I,J,K+2),W
    1 (I,J,K+2),UX(I,J,K+3),WY(I,J,K+3)
  30 CONTINUE
```

```
              K= K+4
         IF ( K - MTP3 )  4110 , 4110 , 4111
4111     IF( KPO) 411, 4119,411
 411     DO 4118 J = JSS, JSS1
         DO 4118 K = 3 , MTP3
               IN1 = I-1
               JN1  = J - 1
               KN1 = K - 2
      PRINT 40,IN1, JN1, KN1, UX(I,J,K), WY(I,J,K)
4118  CONTINUE
4119     DO  399  I = 1 , MXP1
          DO 399  J = 1, JS2
          DO 399  K= 1, 2
             UX(I,J,K) = 0.0
             WY(I,J,K) = 0.0
 399  CONTINUE
             K = 3
8961     DO 145  I = 2, MXP1
               DO  145  J = JSS, JS2
               UX(I,J,K) = UX(I,J,K-1)
                   WY(I,J,K)= WY(I,J,K-1)
 145  CONTINUE
             JS2 = JSS1 + MY
             JSS2 = JSS1 + 1
         IF(  JTEST)  7407, 7408 , 7407
7407  READ 7447 , ITT1, ITT2, JTT1, JTT2
7408     IF( JSS- 1 )   700, 700, 8
   8         JS4 = JSS - 1
          DO 147  J = 1, JS4
          DO 147  I = 2, MXP1
               UX( I,J,K) = 0.0
             WY(I,J,K) = 0.0
 147  CONTINUE
          GO TO 700
 700     IF( KASE) 7001, 7002 , 7001
7001  PRINT 500
7002           JSI =  JS2 - 1
             JS4 = JSS - 1
         IF( JTEST )  7508, 7507, 7508
7508     DO 7409 I = ITT1, ITT2
         DO 7409 J = JTT1, JTT2
               UX(I,J,K) = 0.0
               WY(I,J,K) = 0.0
7409  CONTINUE
7507     GO TO ( 81 , 9 , 81 ) , JB
  81     DO  149  I =  1 , MXP1
             UX(I,1,K) = 2.0 * UX(I,2,K) - UX(I,3,K)
             WY(I,1,K) = 2.0 * WY(I,2,K) - WY(I,3,K)
 149  CONTINUE
         IF (  JSS - 1 )   7 , 7 , 7307
7307     DO 150  J = 2 ,  JS4
             UX(1,J,K) = 2.0 * UX(2,J,K) - UX(3,J,K)
             WY(1,J,K) = 2.0 * WY(2,J,K) - WY(3,J,K)
 150  CONTINUE
         GO TO 7
```

```
    9     DO 148   J= 1, JS4
              UX(1,J,K) = 2.0 * UX(2,J,K) -UX(3,J,K)
              WY(1,J,K) = 2.0 * WY(2,J,K) -WY(3,J,K)
  148 CONTINUE
    7     DO 1490  J = JSS2 , JSI
              UX(1,J,K) = 2.0 * UX(2,J,K) - UX(3,J,K)
              WY(1,J,K) = 2.0 * WY(2,J,K) - WY(3,J,K)
          IF( LB)  9944, 1490, 9944
 9944            UX( I,J,K) = 0.0
                 WY(I,J,K) = 0.0
 1490 CONTINUE
C---------  -  COMPUTATION OF RESIDUALS
          DO 50   I = 2   , MX
           DO 50 J = JTT  , JSI
          IF( JTEST)  7513 , 7412 , 7513
 7513     IF( I - ITT1 )  7412, 7413 , 7416
 7413     IF( J - JTT1 )   7412 , 50, 7415
 7415     IF( J - JTT2 )  50, 50, 7412
 7416     IF( I - ITT2 )  7413, 7413, 7412
 7412            UX( I,J,1 ) = - UX(I,J,3 )
              WY(I,J,1) = -WY(I,J,3)
15012         A = (UX(I-1,J,K) - 2.0 * UX(I,J,K) +UX(I+1,J,K))/(HX**2.0)
          IF( J - JSS1 )  9902, 9902 , 1202
 9902     GO TO (  1202, 1302, 1402, 1502, 1602, 1702, 1802, 1902),KB
 1302              WY( 1,J,K) = 0.15*  WY(2,J,K)
          GO TO 1202
 1402              WY( 1,J,K) =0.3  *  WY(2,J,K)
          GO TO 1202
 1502              WY( 1,J,K) =0.45 *  WY(2,J,K)
          GO TO 1202
 1602              WY( 1,J,K) =0.60 *  WY(2,J,K)
          GO TO 1202
 1702              WY( 1,J,K) =0.75 *  WY(2,J,K)
          GO TO 1202
 1802              WY( 1,J,K) =0.90 *  WY(2,J,K)
          GO TO 1202
 1902              WY( 1,J,K) =          WY(2,J,K)
 1202          D = (WY(I+1,J,K) + WY(I-1,J,K) -2.0 * WY(I,J,K))/(HX**2)
          GO TO (  306, 3484, 307)   ,  JB
  307     IF(  J- JTT ) 3071 , 3071, 3484
 3071     IF( I-2 )  359, 359 , 3435
 3484     IF( J- JS4 )  6344 , 3424 , 4306
 6344     IF( I - 2 )  6345 , 6345 , 344
 4306     IF( J- JSS) 343 , 343 , 348
 6345            F=( -WY(I,J-1,K)+WY(I,J+1,K))/(2.0*HY)
                 G=(-UX(I,J-1,K)+UX(I,J+1,K))/(2.0*HY)
                 B=( UX(I+1,J,K)-UX(I,J,K))/(HX)
                 C=( WY(I+1,J,K)- WY(I,J,K))/(HX)
       L=(UX(I+1,J+1,K)-UX(I,J+1,K)-UX(I+1,J-1,K)+UX(I,J-1,K))/(2*HX*HY
       E=(WY(I+1,J+1,K)-WY(I,J+1,K)-WY(I+1,J-1,K)+WY(I,J-1,K))/(2*HX*HY
                 M=(UX(I,J-1,K)-2.0*UX(I,J,K)+UX(I,J+1,K))/(HY**2)
                NN=(WY(I,J-1,K)-2.0*WY(I,J,K)+WY(I,J+1,K))/(HY**2)
          GO TO 644
 3424     IF( I- 2 ) 3425 , 3425 , 342
 3425            F=( WY(I,J+1,K)-WY(I,J-1,K))/(2.0*HY)
```

```
               G=( UX(I,J,K) - UX(I,J-1,K))/(HY)
               B=( UX(I+1,J,K)- UX(I,J,K))/(HX)
               C=( WY(I+1,J,K)- WY(I,J,K))/(HX)
               M = (UX(I,J-2,K) -2.0*UX(I,J-1,K) + UX(I,J,K))/(HY**2)
              NN = (WY(I,J-2,K) -2.0*WY(I,J-1,K) + WY(I,J,K))/(HY**2)
        E=(WY(I+1,J,K)-WY(I+1,J-1,K)-WY(I,J,K)+WY(I,J-1,K))/(  HX*HY)
        L=(UX(I+1,J,K)-UX(I+1,J-1,K)-UX(I,J,K)+UX(I,J-1,K))/(  HX*HY)
        GO TO 644
306     IF(  J - JSS )   343 , 343 , 348
348     IF(J-JSS1)  344 , 342 , 349
344            F=( WY(I,J+1,K)-WY(I,J-1,K))/(2.0*HY)
               G= (-UX(I,J-1,K)+ UX(I,J+1,K))/(2.0*HY)
               B=(UX(I+1,J,K)- UX(I-1,J,K))/(2.0*HX)
               C=(WY(I+1,J,K)- WY(I-1,J,K))/(2.0*HX)
             L = (UX(I+1,J+1,K) - UX(I-1,J+1,K) +UX(I-1,J-1,K) -UX(I+1,
    1          J-1,K) )/(4.0*HX*HY)
             M = (UX(I,J-1,K) - 2.0*UX(I,J,K) + UX(I,J+1,K))/(HY**2.0)
            NN = (WY(I,J-1,K) - 2.0*WY(I,J,K) + WY(I,J+1,K))/(HY**2.0)
             E = (WY(I+1,J+1,K) - WY(I-1,J+1,K) + WY(I-1,J-1,K) - WY(
    1          I+1,J-1,K)) / (4.0*HX*HY)
        GO TO 644
342            F=( WY(I,J+1,K)- WY(I,J-1,K))/(2.0*HY)
               G=( UX(I,J,K)- UX(I,J-1,K))/(HY)
               B= ( UX(I+1,J,K) - UX(I-1,J,K))/(2.0*HX)
               C= ( WY(I+1,J,K) - WY(I-1,J,K))/(2.0*HX)
     L=(UX(I+1,J,K)-UX(I-1,J,K)-UX(I+1,J-1,K)+UX(I-1,J-1,K))/(2*HX*HY)
               M= (UX(I,J-2,K)-2.0*UX(I,J-1,K)+UX(I,J,K))/(HY**2)
               NN=( WY(I,J-1,K)-2.0*WY(I,J,K)+WY(I,J+1,K))/(HY**2)
      E= (WY(I+1,J+1,K)-WY(I-1,J+1,K)+WY(I-1,J-1,K)-WY(I+1,J-1,K))/
    1          (4.0*HX*HY)
        GO TO 644
349     IF ( J - JSS2 )  1349 , 1349 ,6344
1349    IF (  I - 2 )  359 , 359 , 3435
3435           F=( WY(I,J+1,K)- WY(I,J-1,K))/(2.0*HY)
               G= ( UX(I,J+1,K)- UX(I,J,K))/(HY)
               B= ( UX(I+1,J,K) - UX(I-1,J,K))/(2.0*HX)
               C= ( WY(I+1,J,K) - WY(I-1,J,K))/(2.0*HX)
     L=(UX(I+1,J+1,K)-UX(I-1,J+1,K)-UX(I+1,J,K)+UX(I-1,J,K))/(2*HX*HY)
      E= (WY(I+1,J+1,K)-WY(I-1,J+1,K)+WY(I-1,J-1,K)-WY(I+1,J-1,K))/
    1          (4.0*HX*HY)
               M = (UX(I,J+2,K)-2.0*UX(I,J+1,K)+UX(I,J,K))/(HY**2)
               NN=( WY(I,J-1,K)-2.0*WY(I,J,K)+WY(I,J+1,K))/(HY**2)
        GO TO 644
359            F=( WY(I,J+1,K)-WY(I,J-1,K))/(2.0*HY)
               G=( UX(I,J+1,K)- UX(I,J,K))/(HY)
               B=( UX(I+1,J,K)- UX(I,J,K))/(HX)
               C=( WY(I+1,J,K)- WY(I,J,K))/(HX)
               M=( UX(I,J+2 ,K)-2.0*UX(I,J+1,K)+UX(I,J,K))/(HY**2)
               NN=( WY(I,J-1,K)-2.0*WY(I,J,K)+WY(I,J+1,K))/(HY**2)
     E=(WY(I+1,J+1,K)-WY(I,J+1,K)-WY(I+1,J-1,K)+WY(I,J-1,K))/(2*HX*HY)
     L=(UX(I+1,J+1,K)-UX(I,J+1,K)-UX(I+1,J,K)+UX(I,J,K))/(HX*HY)
        GO TO 644
343            F=( WY(I,J+1,K)-WY(I,J-1,K))/(2.0*HY)
               G= ( UX(I,J+1,K)-UX(I,J,K))/(HY)
               B=(UX(I+1,J,K)- UX(I-1,J,K))/(2.0*HX)
```

```
                 C=(WY(I+1,J,K)- WY(I-1,J,K))/(2.0*HX)
      L=(UX(I+1,J+1,K)-UX(I-1,J+1,K)-UX(I+1,J,K)+UX(I-1,J,K))/(2*HX*HY)
      E=(WY(I+1,J+1,K)-WY(I-1,J+1,K)-WY(I+1,J,K)+WY(I-1,J,K))/(2*HX*HY)
                 M=( UX(I,J+2 ,K)-2.0*UX(I,J+1,K)+UX(I,J,K))/(HY**2)
                NN=( WY(I,J+2 ,K)-2.0*WY(I,J+1,K)+WY(I,J,K))/(HY**2)
 644             PH        = ( UX(I,J,K) - 2.0*UX(I,J,K-1)+ UX(I,J,K-2)) /
     1           (HT**2.0)
            IF( ABSF(PH) - TOLP )  6441, 6441, 6442
6441             PH = 0.0
6442             PS=( WY(I,J,K-2)-2.0*WY(I,J,K-1)+WY(I,J,K))/(HT**2)
            IF( ABSF(PS)  - TOLP)  6443, 6443, 6444
6443             PS= 0.0
6444             EX(I,J,K)= B  -(1.0/2.0)*((B**2)+(C**2))
                EEX= ABSF(EX(I,J,K))
            IF( EEX - TOLP)  9444, 9444, 9445
9444             EEX= 0.0
9445             DDM= AP1+ (AP2*EEX)+( AP3*(EEX**2))+(AP4*(EEX**3))+(AP5*(
     1     EEX**4))+(AP6*(EEX**5))+(AP7*(EEX**6))
            IF( DDM) 4948,4948,4949
4948             DDM=  25.
4949        PNU=CP1+(CP2*EEX)+(CP3*(EEX**2))+(CP4*(EEX**3))+(CP5*(EEX**4))
            IF( PNU)  756, 756, 9875
9875        IF( PNU - 0.49) 755, 756, 756
 756             PNU = 0.49
 755             VEM=( PNU*DDM)/ ((1.0+ PNU)*(1.0-2.0*PNU))
                 SHM =  DDM / ( 2.0*(1.0 + PNU))
                 DM = 2.0* SHM
             VDR = (VEM +DM )
             VSR = (VEM + SHM)
3616        GO TO (  14, 15, 16)   , JB
  14         IF (J-2) 3006, 3006, 3002
3006             REX(I,J,K)= VDR*(A*(1.0-B) - C*D)+VSR*(E*(1.0-F) -G*L)+
     1           SHM*(M*(1.0-B)-NN*C) - PH*RHO
                 REY(I,J,K)=(1.0/HY)*(VDR*(F-(1.0/2.0)*((F**2)+(G**2)))+
     1           VEM*(B -(1.0/2.0)*((B**2)+(C**2)))-STY(I,J-1,K))+SHM*
     2           (L*(1.0-B) -A*G +D*(1.0-F)-E*C) -RHO*(PS)+ PHO
            IF( KTEST)  111, 50,111
3002         IF ( J - JSS1) 808, 808, 809
 808             REX(I,J,K)= VDR*(A*(1.0-B) - C*D)+VSR*(E*(1.0-F) -G*L)+
     1           SHM*(M*(1.0-B)-NN*C) - PH*RHO
                 REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
     1            SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
            IF( KTEST)  111, 50,111
 809         IF ( I-2) 810, 810, 811
 810        IF(  LB)  811, 1810, 811
1810             REX(I,J,K)=( 1.0/HX)*(VDR*(B-(1.0/2.0)*((B**2)+(C**2)))+
     1           VEM*(F-(1.0/2.0)*((F**2)+(G**2))))+SHM*(M*(1.0-B) - L*G+
     2          E*(1.0-F) -NN*C) - RHO*PH
                 REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
     1            SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
            IF( KTEST)  111, 50,111
 811             REX(I,J,K)= VDR*(A*(1.0-B) - C*D)+VSR*(E*(1.0-F) -G*L)+
     1           SHM*(M*(1.0-B)-NN*C) - PH*RHO
                 REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
     1            SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
```

```
      IF( KTEST)  111, 50,111
 15         REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
    1        SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
       IF ( I-2 ) 812, 812, 815
 812   IF (J - JS4) 813, 813,814
 813        REX(I,J,K)=( 1.0/HX)*(VDR*(B-(1.0/2.0)*((B**2)+(C**2)))+
    1       VEM*(F-(1.0/2.0)*((F**2)+(G**2))))+SHM*(M*(1.0-B) - L*G+
    2      E*(1.0-F) -NN*C) - RHO*PH
      IF( KTEST)  111, 50,111
 814   IF (J-JSS1) 815, 815,816
 815        REX(I,J,K)= VDR*(A*(1.0-B) - C*D)+VSR*(E*(1.0-F) -G*L)+
    1       SHM*(M*(1.0-B)-NN*C) - PH*RHO
      IF( KTEST)  111, 50,111
 816  IF( LB)  815, 1816, 815
1816        REX(I,J,K)=( 1.0/HX)*(VDR*(B-(1.0/2.0)*((B**2)+(C**2)))+
    1       VEM*(F-(1.0/2.0)*((F**2)+(G**2))))+SHM*(M*(1.0-B) - L*G+
    2      E*(1.0-F) -NN*C) - RHO*PH
      IF( KTEST)  111, 50,111
 16    IF (I-2)  820,820,821
 820   IF (J-2)  822,822,823
 822        REX(I,J,K)=( 1.0/HX)*(VDR*(B-(1.0/2.0)*((B**2)+(C**2)))+
    1       VEM*(F-(1.0/2.0)*((F**2)+(G**2))))+SHM*(M*(1.0-B) - L*G+
    2      E*(1.0-F) -NN*C) - RHO*PH
            REY(I,J,K)=(1.0/HY)*(VDR*(F-(1.0/2.0)*((F**2)+(G**2)))+
    1       VEM*(B -(1.0/2.0)*((B**2)+(C**2)))-STY(I,J-1,K))+SHM*
    2       (L*(1.0-B) -A*G +D*(1.0-F)-E*C) -RHO*(PS)+ PHO
      IF( KTEST)  111, 50,111
 823   IF (J-JS4) 824,824,825
 824        REX(I,J,K)=( 1.0/HX)*(VDR*(B-(1.0/2.0)*((B**2)+(C**2)))+
    1       VEM*(F-(1.0/2.0)*((F**2)+(G**2))))+SHM*(M*(1.0-B) - L*G+
    2      E*(1.0-F) -NN*C) - RHO*PH
            REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
    1        SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
      IF( KTEST)  111, 50,111
 825   IF ( J - JSS1) 826,826, 827
 826        REX(I,J,K)= VDR*(A*(1.0-B) - C*D)+VSR*(E*(1.0-F) -G*L)+
    1       SHM*(M*(1.0-B)-NN*C) - PH*RHO
            REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
    1        SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
      IF( KTEST)  111, 50,111
 827  IF( LB)  826, 1827, 826
1827        REX(I,J,K)=( 1.0/HX)*(VDR*(B-(1.0/2.0)*((B**2)+(C**2)))+
    1       VEM*(F-(1.0/2.0)*((F**2)+(G**2))))+SHM*(M*(1.0-B) - L*G+
    2      E*(1.0-F) -NN*C) - RHO*PH
            REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
    1        SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
      IF( KTEST)  111, 50,111
 821   IF ( J - 2) 828, 828, 829
 828        REX(I,J,K)= VDR*(A*(1.0-B) - C*D)+VSR*(E*(1.0-F) -G*L)+
    1       SHM*(M*(1.0-B)-NN*C) - PH*RHO
            REY(I,J,K)=(1.0/HY)*(VDR*(F-(1.0/2.0)*((F**2)+(G**2)))+
    1       VEM*(B -(1.0/2.0)*((B**2)+(C**2)))-STY(I,J-1,K))+SHM*
    2       (L*(1.0-B) -A*G +D*(1.0-F)-E*C) -RHO*(PS)+ PHO
      IF( KTEST)  111, 50,111
 829        REX(I,J,K)= VDR*(A*(1.0-B) - C*D)+VSR*(E*(1.0-F) -G*L)+
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    1          SHM*(M*(1.0-B)-NN*C) - PH*RHO
               REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
    1           SHM*(D*(1.0-F)- G*A)-(PJ*RHO)+ PHO
          IF( KTEST)  111, 50,111
  111          REX(I,J,K) = REX(I,J,K) +  PHO
               REY(I,J,K) = REY(I,J,K) - PHO
   50 CONTINUE
          IF( KASE) 5061, 5062, 5061
 5061 PRINT 500
          DO 5063   I  = 2 , MX
          DO 5063   J = JII , JJI
      PRINT 6001,I,J,K,UX(I,J,K),WY(I,J,K),REX(I,J,K),REY(I,J,K)
 5063 CONTINUE
 5062      DO 5005  I= 2, MX
          DO 5005  J = JII  , JJI
 7708     IF( ABSF( REX(I,J,K)) - TOL ) 5005, 5005, 5007
 5005 CONTINUE
           DO 7005   I  = 2, MX
          DO 7005  J = JII  , JJI
          IF( ABSF( REY(I,J,K)) - TOL   )   7005, 7005 , 7009
 7005 CONTINUE
          IF( LTEST) 9262,962,9262
 9262 PRINT 9363
          DO 9464  I  = 2  , MX
          DO 9464  J = JII  , JJI
      PRINT 9565 , I,J,K,REX(I,J,K), REY(I,J,K)
 9464 CONTINUE
  962 PRINT 1212
      PRINT 1054
C---------  -  COMPUTATION OF STRESSES
 1054 FORMAT( 14X,73HCOL ROW K      SIGMAX       SIGMAY             TAU
    1 XDISP      YDISP             )
 5104     DO 5105 I  = 1  , MX
          DO 5105  J = JSS  , JSS1
          GO TO ( 7008, 7099, 7099) , JB
 7008     IF( I  - 1 )  7018, 7018, 7019
 7018     IF( J - JSS)  70120, 70120, 7029
70120          VEM= 0.0
               DM= 0.0
               SHM= 0.0
          GO TO 70125
 7029     IF( J - JSS1 )  7049 , 7049, 70120
 7049          G=  ( UX(I,J,K)- UX(I,J-1,K))/(HY)
 7039          F= ( WY(I,J,K)- WY(I,J-1,K))/(HY)
               B= ( UX(I+1,J,K) - UX(I,J,K)),(HX)
               C= ( WY(I+1,J,K) -  WY(I,J,K))/ ( HX)
               EX(I,J,K)= B -(1.0/2.0)*(B**2  + C**2 )
          GO TO 70123
 7019     IF( J - JSS)  70120, 70120, 70121
70121     IF( J - JSS1) 7022, 7023, 7024
 7022          G= ( -UX(I,J-1,K)+ UX(I,J+1,K))/(2.0*HY)
 7025          B=( -UX(I-1,J,K )+ UX(I+1,J,K))/  ( 2.0*HX)
               C= ( -WY(I-1,J,K) + WY(I+1,J,K))/ ( 2.0*HX)
               F= ( WY(I,J+1,K) - WY(I,J-1,K))/  ( 2.0*HY)
          GO TO 70123
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 7023          G= ( UX(I,J,K) - UX(I,J-1,K))/(HY)
        GO TO 7025
 7024   IF( J - JSS2 )  7124, 7124, 7022
 7124          G= ( UX(I,J+1,K) - UX(I,J,K))/(HY)
        GO TO 7025
 7099   IF( I-1 )  7090, 7090, 7089
 7090   IF( J - JS4 )  70120, 70120 , 7092
 7092   IF( J - JSS)  7091, 7091, 7029
 7091   G= ( UX(I,J+1,K) - UX(I,J,K))/(HY)
        GO TO 7039
 7089   IF( J - JS4 )  70121, 7023 , 7098
 7098   IF( J - JSS)  7124, 7124 , 70121
70123          EEX= ABSF( EX(I,J,K))
               DDM= AP1 +(AP2*EEX)+(AP3*(EEX**2))+(AP4*(EEX**3))+(AP5*
     1  EEX**4))+(AP6*(EEX**5))+(AP7*(EEX**6))
        IF( DDM) 4944,4944,4945
 4944          DDM=  25.
 4945   PNU=CP1+(CP2*EEX)+(CP3*(EEX**2))+(CP4*(EEX**3))+(CP5*(EEX**4))
        IF(  PNU )  758, 758, 1758
 1758   IF( PNU - 0.49)  757, 758, 758
  758          PNU = 0.49
  757          VEM=( PNU*DDM)/ ((1.0+ PNU)*(1.0-2.0*PNU))
               SHM =  DDM / ( 2.0*(1.0 + PNU))
               DM = 2.0* SHM
          VDR = (VEM +DM )
                      VSR = (VEM + SHM)
70125          EY= F - (1.0/2.0)*( F**2  +  G**2 )
               EXY=( G+C)-(G*B)-(F*C)
               SIGMAY          =  VEM*(EX(I,J,K)+EY        ) + DM*EY
               REX(I,J,K)=VEM*(EX(I,J,K) +EY) + DM*EX(I,J,K)
               TAU          =  SHM * EXY
      PRINT 106,I,J,K,REX(I,J,K),SIGMAY,TAU,UX(I,J,K),WY(I,J,K)
  106 FORMAT((15X,I2,1X,I2,1X,I2,2X,E10.3,2X,E10.3,3X,E10.3,2X,E10.3,2X,E
     110.3    )
 5105 CONTINUE
                   I = 1
                    JSS3 = JSS1 - 1
      PRINT  6104
 6104 FORMAT(10X,77HTOTAL HORIZ. FORCE ON PLATE (LB)       TIME(SEC)
     1 MOVEMENT OF THE PLATE       )
 6107          SUM = 0.0
        DO 6103 J = JSS, JSS3
               SUM= SUM+((REX(I,J,K)+ REX(I,J+1,K))/(2.0))*HY
 6103 CONTINUE
               SUMF=   SUM*WD
               TT =(K - 2 ) * HT
      PRINT  6106 , SUMF, TT, UX(1,1,K)
 6106 FORMAT(  25X,E10.3,15X,E10.3,15X,E10.3)
        IF( SUMF  - SUMT )  6701, 1010,    1010
 6701          SUMT = SUMF
             K = K + 1
        IF ( K- MTP3) 961 , 961, 1111
  961   IF( ITEST  ) 8962, 8961, 8962
 8962 READ 240, MY , MMY
        JSS = MMY +1
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         JSS1 =  JSS + MY
         JS2 = JSS1 + MMY
         JS4 =  JSS- 1
         JSI = JS2 - 1
         JTT = 2
                I = 1
         DO 2119 J = JSS, JSS1
      READ 1277  , UX(I,J,K) , WY(I,J,K)
 2119 CONTINUE
         GO TO 8961
 1111 PRINT 1212
         GO TO 1010
 7009           II = I $ JI = J
         GO TO 5009
 5007           FDX =  0.1 * ABSF(REX(I,J,K))
                IT = I  $ JT = J
                MN  = 1.0
                ITER= 1.0
         IF( KOLE)  6101, 60, 6101
 6101 PRINT  681
C---------      LIQUIDATION OF  X  RESIDUALS
   60    DO 51 I = IT , MX
          DO 51  J = JI, JSI
                IN= I+1
                IN1 = I-1
                JN = J +1
                JN1 = J - 1
                EEX= ABSF( EX(I,J,K))
                DDM= AP1+ (AP2*EEX)+( AP3*(EEX**2))+(AP4*(EEX**3))+(AP5*(
     1   EEX**4))+(AP6*(EEX**5))+(AP7*(EEX**6))
         IF( DDM) 4946,4946,4947
 4946           DDM=  25.
 4947    PNU=CP1+(CP2*EEX)+(CP3*(EEX**2))+(CP4*(EEX**3))+(CP5*(EEX**4))
         IF(  PNU )  760, 760, 1760
 1760    IF( PNU - 0.49)  759, 760, 760
  760           PNU = 0.49
  759           VEM=( PNU*DDM)/ ((1.0+ PNU)*(1.0-2.0*PNU))
                SHM =  DDM / ( 2.0*(1.0 + PNU))
                DM = 2.0* SHM
          VSR = (VEM + SHM)
          VDR = (VEM +DM  )
         IF( JTEST)  5016 , 5011, 5016
 5016    IF( I -ITT1)  5011, 6413,6416
 6413    IF( J - JTT1)  5011, 51, 6415
 6415    IF( J - JTT2)  51, 51, 5011
 6416    IF( I - ITT2 )  6413, 6413, 5011
 5011    IF(  K - 3 )  5501, 5501 , 5502
 5501           DELPX= - REX(I,J,K)/((-2.0*VDR)*((1.0/(HX**2))+(SHM/(VDR*
     1          (HY**2)))))
                DELRX= DELPX*(( -2.0*VDR)*((1.0/(HX**2))+(SHM/(VDR*(HY
     1          **2)))))
         GO  TO  5014
 5502           DELPX= - REX(I,J,K)/((-2.0*VDR)*((1.0/(HX**2))+(SHM/(VDR*
     1          (HY**2))))  - (RHO/(HT**2))    )
 5088           DELRX= DELPX*(( -2.0*VDR)*((1.0/(HX**2))+(SHM/(VDR*(HY
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    1          **2)))) - (RHO/(H1**2))    )
5014           UX(I,J,K) = UX(I,J,K) + DELPX
2259      IF( JB - 1 ) 2257, 2257, 2258
2258      IF(  J - JS4 )  2454, 2454, 2257
2257      IF( J - JSS1 )  2256, 2256 , 2454
2256      IF(  ABSF( UX(I,J,K))  -ABSF( UX(I-1,J,K)) )2454, 2454,  2255
2255           REX(I,J,K)= 0.5* REX(I,J,K)
               UX(I,J,K) =  UX(I,J,K) - DELPX
          IF( ABSF( REX(I,J,K)) - TOL )  2454, 2454, 5011
2454           REX(I,J,K) = REX(I,J,K) + DELRX
           IF (I-2) 53, 53, 55
  53           DELRX1  =DELPX*VDR*((1.0/(HX**2))- 1.0/(2.0*(HX**3)))
               REX(IN,J,K)= REX(IN,J,K) + DELRX1
          GO TO 54
  55       IF (I - MX) 555, 556, 556
 555           DELRX1  =DELPX*VDR*((1.0/(HX**2))- 1.0/(2.0*(HX**3)))
               REX(IN,J,K)= REX(IN,J,K) + DELRX1
 556           DELRX2  =DELPX*VDR*((1.0/(HX**2))+ 1.0/(2.0*(HX**3)))
               REX(IN1,J,K)= REX(IN1,J,K) + DELRX2
          GO TO 54
  54      IF (  J - 2 )   56 , 56 , 57
  56           DELRX3 =  DELPX* SHM*( 1.0/(HY**2))
               REX(I,JN,K)= REX(I,JN,K) + DELRX3
          GO TO 51
 57       IF ( J -JSI )  566 , 58 , 51
 566           DELRX3 =  DELPX* SHM*( 1.0/(HY**2))
               REX(I,JN,K)= REX(I,JN,K) + DELRX3
  58           DELRX4 =  DELPX* SHM*( 1.0/(HY**2))
               REX(I,JN1,K)= REX(I,JN1,K) + DELRX4
          GO TO 51
  51 CONTINUE
           DO 511  I = 2, MX
          DO 511 J = JII, JSI
          IF( ABSF(   REX(I,J,K)) - FDX  ) 512 , 512 , 513
 513           II = I
               JT = J
               ITER = ITER +1
          IF( ITER - IM)  60, 60, 512
 512      IF( KOLE)  5002, 511, 5002
5002 PRINT 68, I,J,K, REX(I,J,K), ITER,  UX(I,J,K)
 511  CONTINUE
          GO TO ( 600 , 601 ,602 ) , JB
 600       DO 249  I = 1,MXP1
             UX(I,1,K) = 2.0 * UX(I,2,K) - UX(I,3,K)
             WY(I,1,K) = 2.0 * WY(I,2,K) - WY(I,3,K)
 249 CONTINUE
           IF (JSS-1) 602, 602, 603
 603       DO 604  J = 2, JS4
             UX(1,J,K) = 2.0 * UX(2,J,K) - UX(3,J,K)
             WY(1,J,K) = 2.0 * WY(2,J,K) - WY(3,J,K)
 604 CONTINUE
           GO TO 602
 601      DO  3448 J = 1 , JS4
             UX(1,J,K) = 2.0 * UX(2,J,K) - UX(3,J,K)
             UX(1,J,K) = 2.0 * UX(2,J,K) - UX(3,J,K)
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3448 CONTINUE
 602     DO 3449 J = JSS2 , JSI
              WY(1,J,K) = 2.0 * WY(2,J,K) - WY(3,J,K)
                 UX(1,J,K)= 2.0*UX(2,J,K)-UX(3,J,K)
         IF ( LB )  9884, 3449, 9884
9884           UX( I,J,K) = 0.0
               WY( I,J,K) = 0.0
3449 CONTINUE
C---------     COMPUTATION OF NEW  Y  RESIDUALS
          DO 507  I = 2, MX
         DO 507  J = JTI , JSI
               EEX= ABSF( EX(I,J,K))
8445           DDM= AP1+ (AP2*EEX)+( AP3*(EEX**2))+(AP4*(EEX**3))+(AP5*(
    1    EEX**4))+(AP6*(EEX**5))+(AP7*(EEX**6))
         IF( DDM) 4940,4940,4941
4940           DDM=  25.
4941     PNU=CP1+(CP2*EEX)+(CP3*(EEX**2))+(CP4*(EEX**3))+(CP5*(EEX**4))
         IF( PNU) 762, 762, 1762
1762     IF( PNU  -  0.49)  761, 761, 762
 762           PNU = 0.49
 761           VEM=( PNU*DDM)/ ((1.0+ PNU)*(1.0-2.0*PNU))
               SHM =  DDM / ( 2.0*(1.0 + PNU))
               DM = 2.0* SHM
          VDR = (VEM +DM )
          VSR = (VEM + SHM)
         IF( JTEST)  1012 , 8412 , 1012
1012     IF( I -ITT1)  8412, 8413, 8416
8413     IF( J - JTT1)  8412, 507, 8415
8415     IF( J - JTT2)  507, 507, 8412
8416     IF( I - ITT2 )  8413, 8413    ,  8412
8412         UX(I,J,1) = -UX(I,J,3)
             WY(I,J,1) = -WY(I,J,3)
16012        A = (UX(I-1,J,K) - 2.0 * UX(I,J,K) +UX(I+1,J,K))/(HX**2.0)
         IF( J - JSS1 ) 8802, 8802, 4102
8802     GO TO (  4102, 4202, 4302, 4402, 4502, 4602, 4702,4802), KB
4202           WY( 1,J,K) = 0.15*  WY(2,J,K)
         GO TO 4102
4302           WY( 1,J,K) =0.3   *  WY(2,J,K)
         GO TO 4102
4402           WY( 1,J,K) =0.45 *  WY(2,J,K)
         GO TO 4102
4502           WY( 1,J,K) =0.60 *  WY(2,J,K)
         GO TO 4102
4602           WY( 1,J,K) =0.75 *  WY(2,J,K)
         GO TO 4102
4702           WY( 1,J,K) =0.90 *  WY(2,J,K)
         GO TO 4102
4802           WY( 1,J,K) =         WY(2,J,K)
4102         D = (WY(I+1,J,K) + WY(I-1,J,K) -2.0 * WY(I,J,K))/(HX**2)
1403     GO TO ( 406, 5484, 407 ) , JB
 407     IF( J - JII )  4071 , 4071 , 5484
4071     IF ( I - 2 )  559 , 559 , 5435
5484     IF( J - JS4 )  7344 , 5424 ,  6406
6406     IF( J - JSS )  543 , 543 , 548
7344     IF (  I - 2 )  7345 ,  7345 ,  544
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 7345           F=( -WY(I,J-1,K)+WY(I,J+1,K))/(2.0*HY)
                G=(-UX(I,J-1,K)+UX(I,J+1,K))/(2.0*HY)
                B=( UX(I+1,J,K)-UX(I,J,K))/(HX)
                C=( WY(I+1,J,K)- WY(I,J,K))/(HX)
      L=(UX(I+1,J+1,K)-UX(I,J+1,K)-UX(I+1,J-1,K)+UX(I,J-1,K))/(2*HX*HY)
      E=(WY(I+1,J+1,K)-WY(I,J+1,K)-WY(I+1,J-1,K)+WY(I,J-1,K))/(2*HX*HY)
                M=(UX(I,J-1,K)-2.0*UX(I,J,K)+UX(I,J+1,K))/(HY**2)
               NN=(WY(I,J-1,K)-2.0*WY(I,J,K)+WY(I,J+1,K))/(HY**2)
       GO TO 744
 5424   IF(I-2)  5425 , 5425 , 542
 5425           F=( WY(I,J+1,K)-WY(I,J-1,K))/(2.0*HY)
                G=( UX(I,J,K) - UX(I,J-1,K))/(HY)
                B=( UX(I+1,J,K)- UX(I,J,K))/(HX)
                C=( WY(I+1,J,K)- WY(I,J,K))/(HX)
        L = ( UX(I+1,J,K)-UX(I+1,J-1,K)-UX(I,J,K)+UX(I,J-1,K))/(HX*HY)
                M = (UX(I,J-2,K) -2.0*UX(I,J-1,K) + UX(I,J,K))/(HY**2)
               NN = (WY(I,J-2,K) -2.0*WY(I,J-1,K) + WY(I,J,K))/(HY**2)
        E = ( WY(I+1,J,K)-WY(I+1,J-1,K)-WY(I,J,K)+WY(I,J-1,K))/(HX*HY)
                GO TO 744
  406   IF( J - JSS )  543 , 543 , 548
  548   IF(J-JSS1) 544,542,549
  544           F=( WY(I,J+1,K)-WY(I,J-1,K))/(2.0*HY)
                G= (-UX(I,J-1,K)+ UX(I,J+1,K))/(2.0*HY)
                B=(UX(I+1,J,K)- UX(I-1,J,K))/(2.0*HX)
                C=(WY(I+1,J,K)- WY(I-1,J,K))/(2.0*HX)
              L = (UX(I+1,J+1,K) - UX(I-1,J+1,K) +UX(I-1,J-1,K) -UX(I+1,
     1          J-1,K) )/(4.0*HX*HY)
              M = (UX(I,J-1,K) - 2.0*UX(I,J,K) + UX(I,J+1,K))/(HY**2.0)
             NN = (WY(I,J-1,K) - 2.0*WY(I,J,K) + WY(I,J+1,K))/(HY**2.0)
              E = (WY(I+1,J+1,K) - WY(I-1,J+1,K) + WY(I-1,J-1,K) - WY(
     1          I+1,J-1,K)) / (4.0*HX*HY)
        GO TO 744
  542           F=( WY(I,J+1,K)- WY(I,J-1,K))/(2.0*HY)
                G=( UX(I,J,K)- UX(I,J-1,K))/(HY)
                B= ( UX(I+1,J,K) - UX(I-1,J,K))/(2.0*HX)
                C= ( WY(I+1,J,K) - WY(I-1,J,K))/(2.0*HX)
      L=(UX(I+1,J,K)-UX(I-1,J,K)-UX(I+1,J-1,K)+UX(I-1,J-1,K))/(2*HX*HY)
                M= (UX(I,J-2,K)-2.0*UX(I,J-1,K)+UX(I,J,K))/(HY**2)
                NN=( WY(I,J-1,K)-2.0*WY(I,J,K)+WY(I,J+1,K))/(HY**2)
       E= (WY(I+1,J+1,K)-WY(I-1,J+1,K)+WY(I-1,J-1,K)-WY(I+1,J-1,K))/
     1          (4.0*HX*HY)
        GO TO 744
  549   IF( J-JSS2 ) 1549,1549, 7344
 1549   IF ( I- 2 )  559 , 559 , 5435
 5435           F=( WY(I,J+1,K)- WY(I,J-1,K))/(2.0*HY)
                G= ( UX(I,J+1,K)- UX(I,J,K))/(HY)
                B= ( UX(I+1,J,K) - UX(I-1,J,K))/(2.0*HX)
                C= ( WY(I+1,J,K) - WY(I-1,J,K))/(2.0*HX)
       E= (WY(I+1,J+1,K)-WY(I-1,J+1,K)+WY(I-1,J-1,K)-WY(I+1,J-1,K))/
     1          (4.0*HX*HY)
      L=(UX(I+1,J+1,K)-UX(I-1,J+1,K)-UX(I+1,J,K)+UX(I-1,J,K))/(2*HX*HY)
                M = (UX(I,J+2,K)-2.0*UX(I,J+1,K)+UX(I,J,K))/(HY**2)
                NN=( WY(I,J-1,K)-2.0*WY(I,J,K)+WY(I,J+1,K))/(HY**2)
        GO TO 744
  559           F=( WY(I,J+1,K)-WY(I,J-1,K))/(2.0*HY)
```
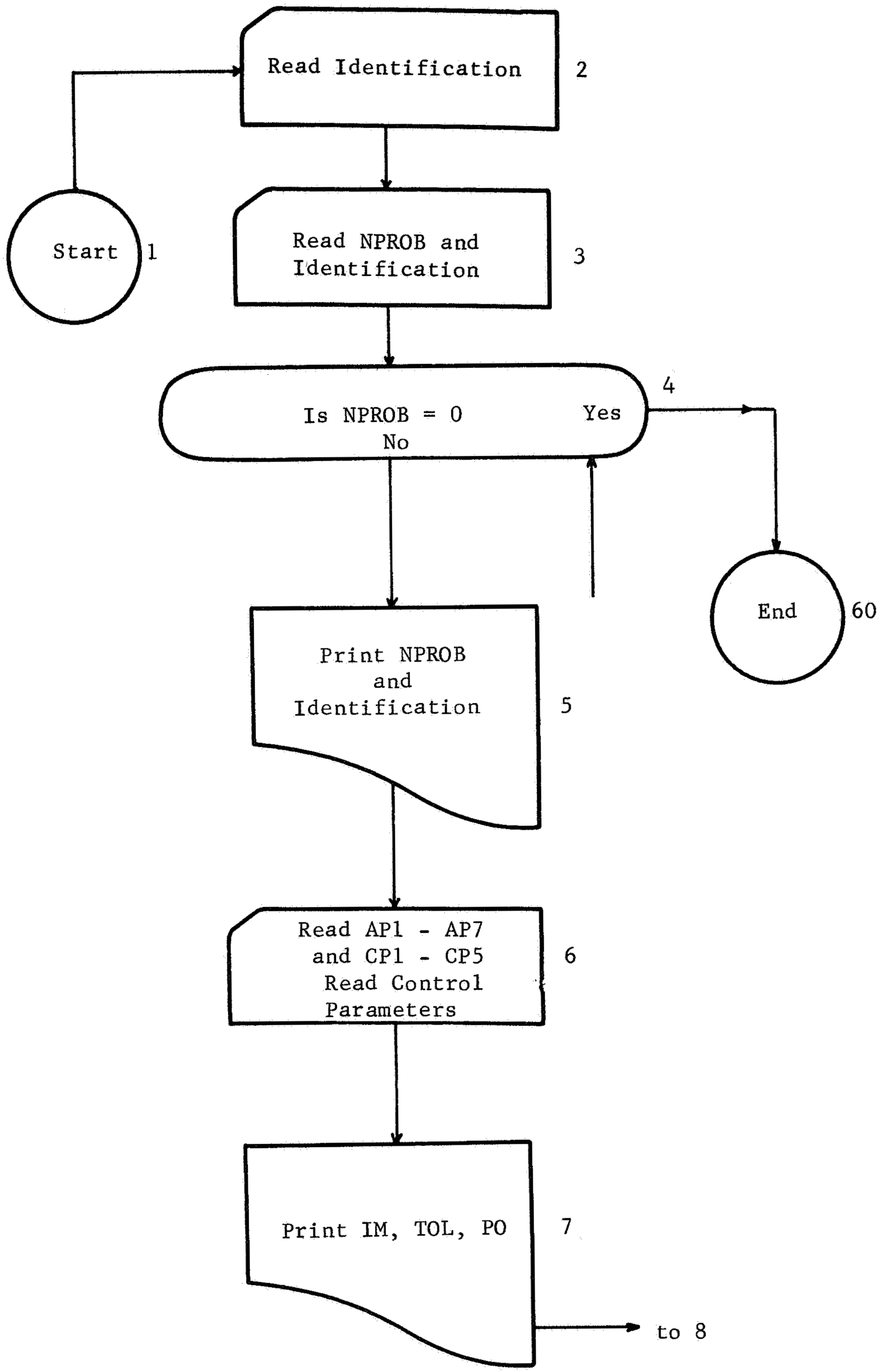
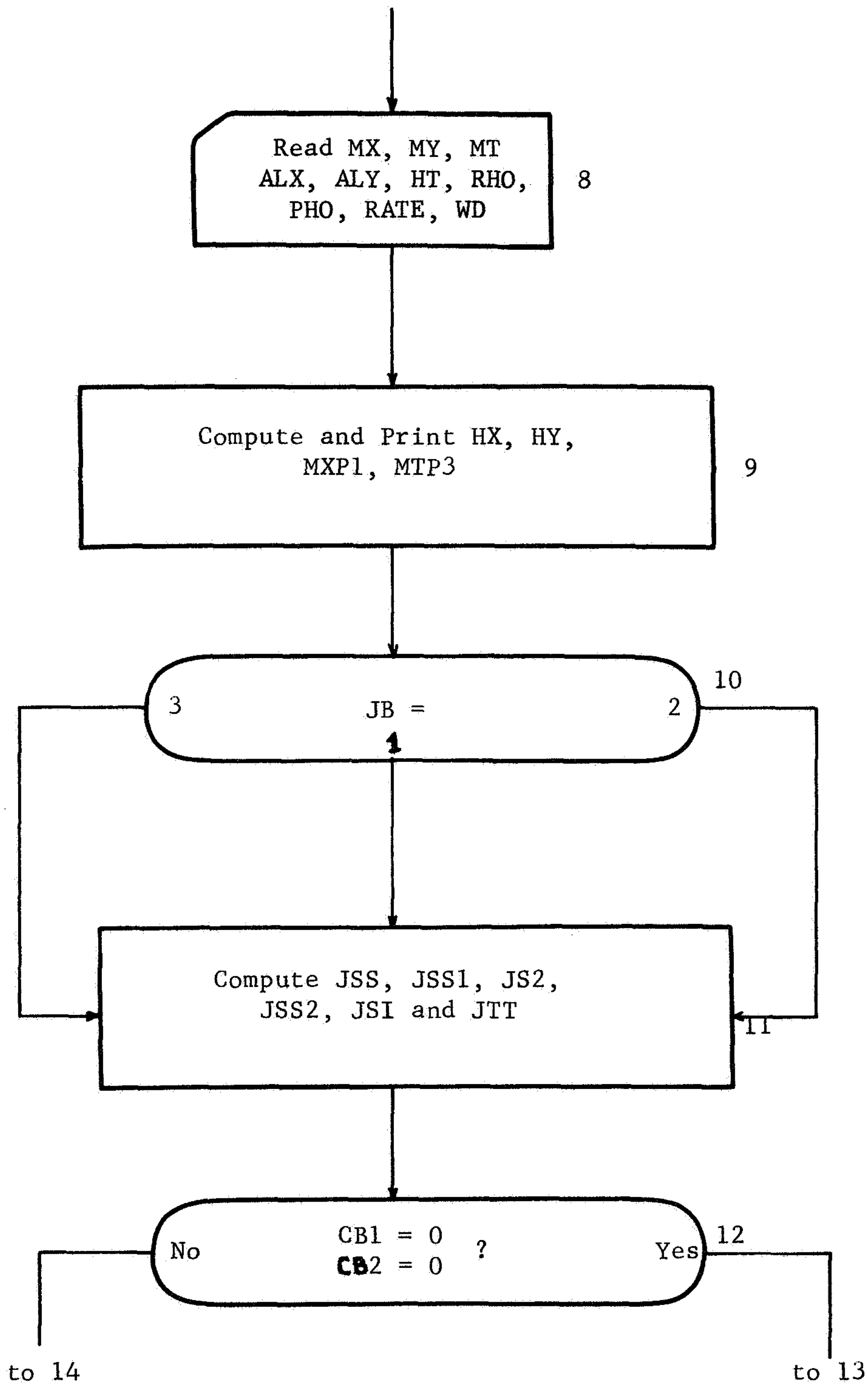

```
              G=( UX(I,J+1,K)- UX(I,J,K))/(HY)
              B=( UX(I+1,J,K)- UX(I,J,K))/(HX)
              C=( WY(I+1,J,K)- WY(I,J,K))/(HX)
              M=( UX(I,J+2 ,K)-2.0*UX(I,J+1,K)+UX(I,J,K))/(HY**2)
              NN=( WY(I,J-1,K)-2.0*WY(I,J,K)+WY(I,J+1,K))/(HY**2)
      L=(UX(I+1,J+1,K)-UX(I,J+1,K)-UX(I+1,J,K)+UX(I,J,K))/(HX*HY)
      E=(WY(I+1,J+1,K)-WY(I,J+1,K)-WY(I+1,J-1,K)+WY(I,J-1,K))/(2*HX*HY)
         GO TO 744
 543          F=( WY(I,J+1,K)-WY(I,J-1,K))/(2.0*HY)
              G= ( UX(I,J+1,K)-UX(I,J,K))/(HY)
              B=(UX(I+1,J,K)- UX(I-1,J,K))/(2.0*HX)
              C=(WY(I+1,J,K)- WY(I-1,J,K))/(2.0*HX)
      L=(UX(I+1,J+1,K)-UX(I-1,J+1,K)-UX(I+1,J,K)+UX(I-1,J,K))/(2*HX*HY)
      E=(WY(I+1,J+1,K)-WY(I-1,J+1,K)-WY(I+1,J,K)+WY(I-1,J,K))/(2*HX*HY)
              M=( UX(I,J+2 ,K)-2.0*UX(I,J+1,K)+UX(I,J,K))/(HY**2)
             NN=( WY(I,J+2 ,K)-2.0*WY(I,J+1,K)+WY(I,J,K))/(HY**2)
 744          PH        = ( UX(I,J,K) - 2.0*UX(I,J,K-1)+ UX(I,J,K-2)) /
    1         (HT**2.0)
         IF( ABSF(PH) - TOLP )  7441, 7441, 7442
7441          PH = 0.0
7442          PS=( WY(I,J,K-2)-2.0*WY(I,J,K-1)+WY(I,J,K))/(HT**2)
         IF( ABSF(PS)  - TOLP)  7443, 7443, 7444
7443          PS= 0.0
7444     GO TO (  830, 831, 832) , JB
 830      IF (J-2)  4006,4006,4002
4006          REY(I,J,K)=(1.0/HY)*(VDR*(F-(1.0/2.0)*((F**2)+(G**2)))+
    1         VEM*(B -(1.0/2.0)*((B**2)+(C**2)))-STY(I,J-1,K))+SHM*
    2         (L*(1.0-B) -A*G +D*(1.0-F)-E*C) -RHO*(PS)+ PHO
         IF( KTEST)  222, 507, 222
4002          REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
    1          SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
         IF( KTEST)  222, 507, 222
 831          REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
    1          SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
         IF( KTEST)  222, 507, 222
 832     IF( J - 2 )  844, 844, 845
 844          REY(I,J,K)=(1.0/HY)*(VDR*(F-(1.0/2.0)*((F**2)+(G**2)))+
    1         VEM*(B -(1.0/2.0)*((B**2)+(C**2)))-STY(I,J-1,K))+SHM*
    2         (L*(1.0-B) -A*G +D*(1.0-F)-E*C) -RHO*(PS)+ PHO
         IF( KTEST)  222, 507, 222
 845          REY(I,J,K)=VDR*(NN*(1.0-F)-G*M)+VSR*(L*(1.0-B)-C*E)+
    1          SHM*(D*(1.0-F)- G*A)-(PS*RHO)+ PHO
         IF( KTEST)  222, 507, 222
 222          REX(I,J,K)  = REX(I,J,K)  +  PHO
              REY(I,J,K)  = REY(I,J,K)  - PHO
 507 CONTINUE
              FDX   = 0.1 * ABSF(REX(I,J,K))
              IT = 2   $ JT = 2
5009    FDY =   0.5 * ABSF ( REY(2,2,K ))
         IF( KOLE)  7201, 720, 7201
7201 PRINT 901
C---------        LIQUIDATION OF  Y  RESIDUALS
 720          ITER = 1.0
  80      DO 72  I = IT, MX
          DO 72 J = JT, JSI
```

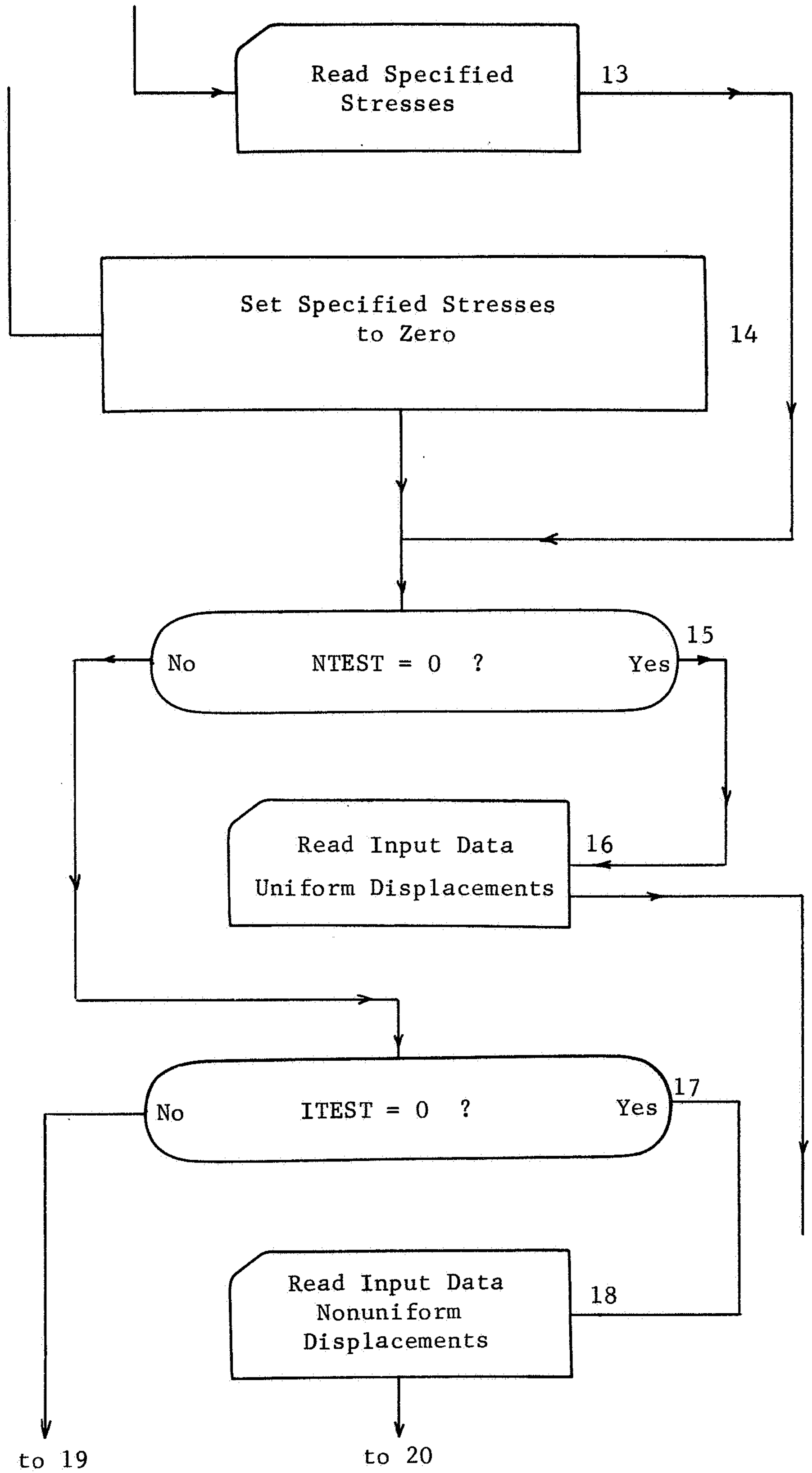
```
            IN= I+1
            IN1 = I - 1
            JN= J +1
            JN1  = J - 1
           EEX= ABSF(EX(I,J,K))
            DDM= AP1+ (AP2*EEX)+( AP3*(EEX**2))+(AP4*(EEX**3))+(AP5*(
     1  EEX**4))+(AP6*(EEX**5))+(AP7*(EEX**6))
       IF( DDM) 4942,4942,4943
4942        DDM= 25.
4943   PNU=CP1+(CP2*EEX)+(CP3*(EEX**2))+(CP4*(EEX**3))+(CP5*(EEX**4))
       IF(  PNU )  764, 764, 1764
1764   IF( PNU - 0.49)  763, 763, 764
 764        PNU = 0.49
 763        VEM=( PNU*DDM)/ ((1.0+ PNU)*(1.0-2.0*PNU))
            SHM =  DDM / ( 2.0*(1.0 + PNU))
            DM = 2.0* SHM
        VDR = (VEM +DM )
        VSR = (VEM + SHM)
       IF( JTEST)  8724,  724, 8724
8724   IF( I - ITT1 ) 724, 9413, 9416
9413   IF( J - JTT1)  724,  72,  9415
9415   IF( J - JTT2 ) 72, 72 , 724
9416   IF( I - ITT2 )  9413, 9413 , 724
 724   IF(  K - 3 )  7724, 7724 , 7725
7724               DELPY        =(     - REY(I,J,K)) /((-2.0*VDR)*((1.0/
     1       (HY**2 ))+(SHM/(VDR*(HX**2)))))
                        DELRY         =DELPY         * ((-2.0*VDR)*((1.0/
     1         (HY**2))+(SHM/(VDR*(HX**2)))))
       GO TO  727
7725               DELPY        =(     - REY(I,J,K)) /((-2.0*VDR)*((1.0/
     1       (HY**2 ))+(SHM/(VDR*(HX**2))))  -(RHO/ (HT**2))    )
                        DELRY         =DELPY         * ((-2.0*VDR)*((1.0/
     1         (HY**2))+(SHM/(VDR*(HX**2))))  - ( RHO/(HT**2))    )
 727        WY(I,J,K) = WY(I,J,K) + DELPY
3357   IF(  WY(I,J,K)) 3358, 3352, 3352
3358   IF( J - JSS1 )  3355, 3355, 3352
3355   IF( ABSF( WY(I,J,K))- ABSF( WY(I,J-1,K)))   3352, 3352, 3351
3351        REY(I,J,K)=  0.5* REY(I,J,K)
            WY(I,J,K)= WY(I,J,K) - DELPY
       IF( ABSF( REY(I,J,K)) - TOL )  3352, 724,   724
3352        REY(I,J,K)= REY(I,J,K) + DELRY
       IF ( I - 2 )  73 , 73 , 75
  73        DELRY1 = DELPY*SHM*(1.0/(HX**2))
            REY(IN ,J,K)= REY(IN ,J,K) + DELRY1
     GO TO 74
  75    IF (I-MX) 733, 79, 79
 733        DELRY1 = DELPY*SHM*(1.0/(HX**2))
            REY(IN ,J,K)= REY(IN ,J,K) + DELRY1
  79        DELRY2 = DELPY*SHM*(1.0/(HX**2))
            REY(IN1,J,K)= REY(IN1,J,K) + DELRY2
        GO TO 74
  74   IF ( J -2 )  76 , 76 , 77
  76        DELRY3 = DELPY*VDR*((1.0/(HY**2))-1.0/(2.0*(HY**3)))
            REY(I,JN   ,K) = REY(I,JN   ,K) + DELRY3
     GO TO 72
```

```
  77     IF ( J - JS I )  797 , 78 , 72
 797           DELRY3 = DELPY*VDR*((1.0/(HY**2))-1.0/(2.0*(HY**3)))
               REY(I,JN  ,K) = REY(I,JN  ,K) + DELRY3
  78           DELRY4 = DELPY*VDR*((1.0/(HY**2))+1.0/(2.0*(HY**3)))
               REY(I,JN1 ,K) = REY(I,JN1 ,K) + DELRY4
  72 CONTINUE
         DO 85  I= 2,MX
         DO  85 J = JTT, JSI
         IF(  ABSF(  REY(I,J,K))  -  FDY ) 855 , 855 , 87
  87           IT=I
               JT = J
               ITER = ITER +1
         IF( ITER - IM)  89, 89 , 700
  89     GO TO 80
  90 PRINT 94
          GO TO 999
 855     IF( KOLE)  5003, 85, 5003
5003 PRINT 92 , I,J,K,  REY(I,J,K),  WY(I,J,K)
  85 CONTINUE
         GO TO 700
 999 CONTINUE
     END
```

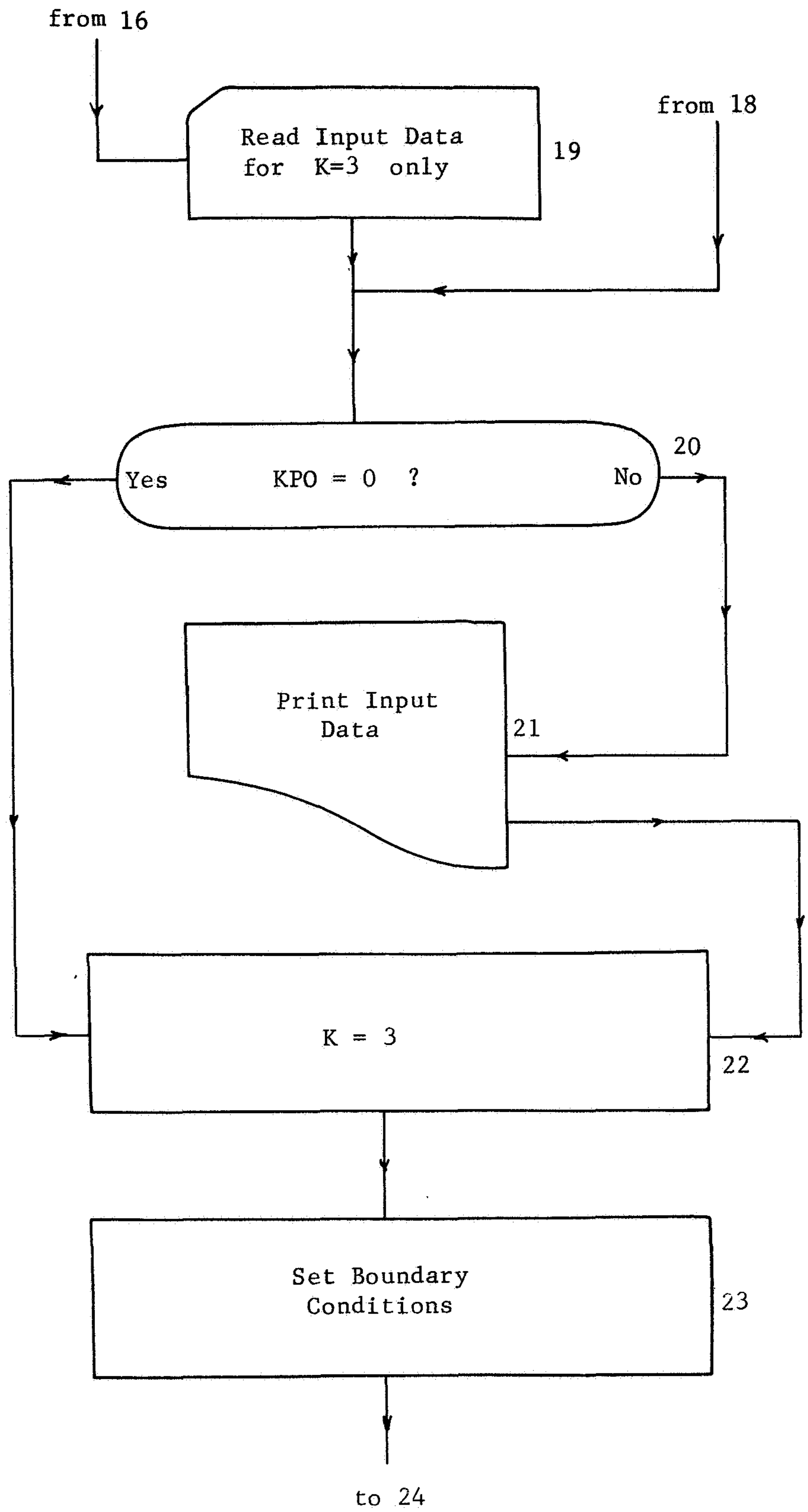
```
      PROGRAM  NASA1   ,  CE0511159

NA12       PLAIN  STRAIN  DYNAMIC  PROBLEM
           27139E+02-27222E+04 13213E+06-32679E+07 41633E+08-25851E+09 613
           23875E-02 46742E+00-19770E+02 29880E+03-14841E+04
                                          20     1     1          3 E+00  1
      12     6    47 54    E+00 18    E+00  5  E-04    15E-02    56 E-01  26
    12   E+00
 133 E-02           266 E-02            399 E-02            532 E-02
 665 E-02           798 E-02            931 E-02           1064 E-02
1197 E-02          1330 E-02           1463 E-02           1596 E-02
1729 E-02          1862 E-02           1995 E-02           2128 E-02
2260 E-02          2393 E-02           2526 E-02           2659 E-02
2792 E-02          2925 E-02           3058 E-02           3191 E-02
3324 E-02          3457 E-02           3590 E-02           3723 E-02
3856 E-02          3989 E-02           4122 E-02           4255 E-02
4388 E-02          4521 E-02           4654 E-02           4787 E-02
4920 E-02          5053 E-02           5186 E-02           5319 E-02
5452 E-02          5585 E-02           5718 E-02           5851 E-02
5984 E-02          6117 E-02           6250 E-02           6383 E-02
```


Read Identification
2
Start
1
Read NPROB and Identification
3
Is NPROB = 0
Yes
No
4
End
60
Print NPROB and Identification
5
Read AP1 - AP7 and CP1 - CP5 Read Control Parameters
6
Print IM, TOL, PO
7
to 8

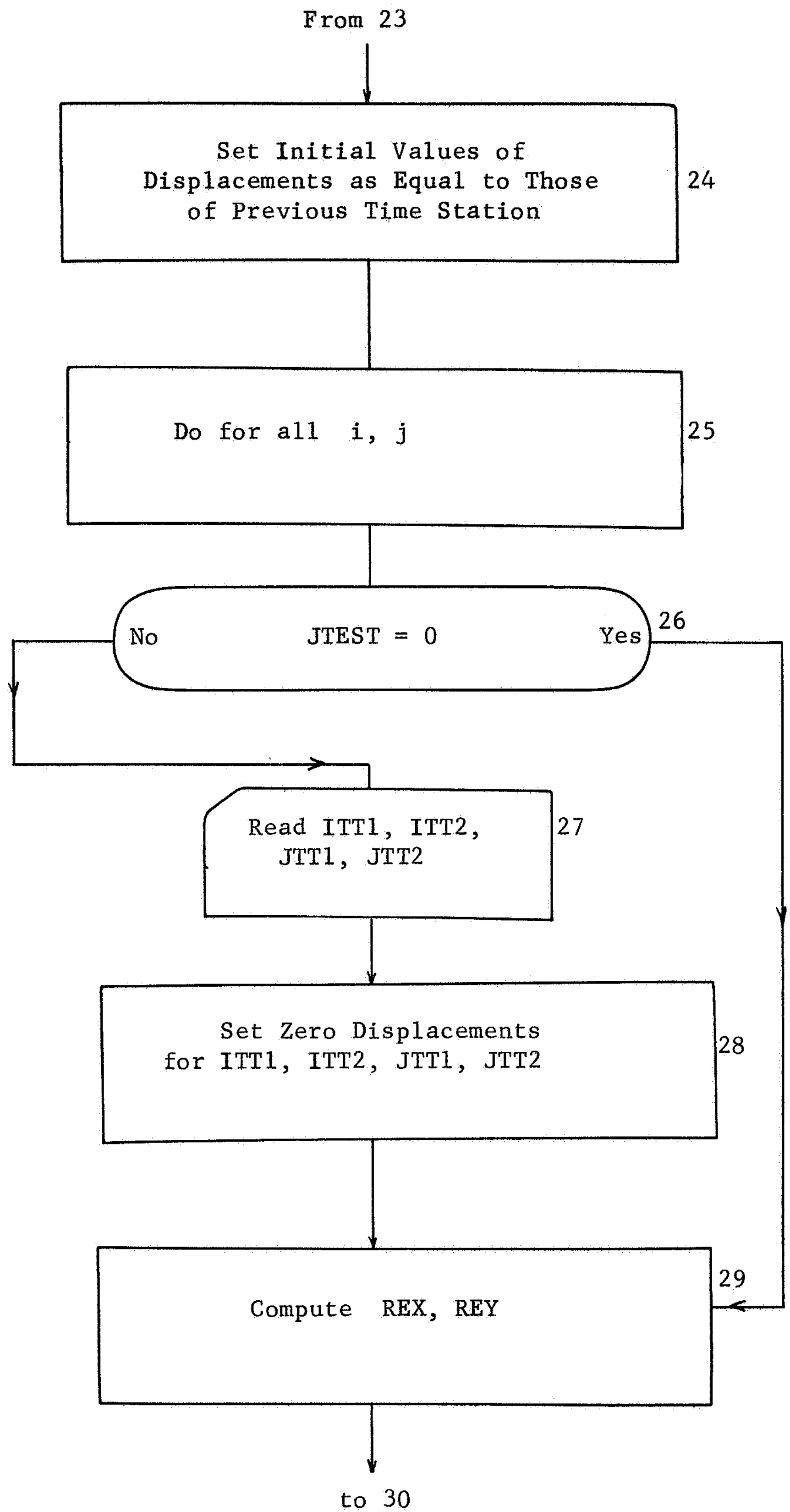

Read MX, MY, MT
ALX, ALY, HT, RHO,
PHO, RATE, WD
8
Compute and Print HX, HY,
MXP1, MTP3
9
3
JB =
2
1
10
Compute JSS, JSS1, JS2,
JSS2, JSI and JTT
11
No
CB1 = 0
CB2 = 0
?
Yes
12
to 14
to 13

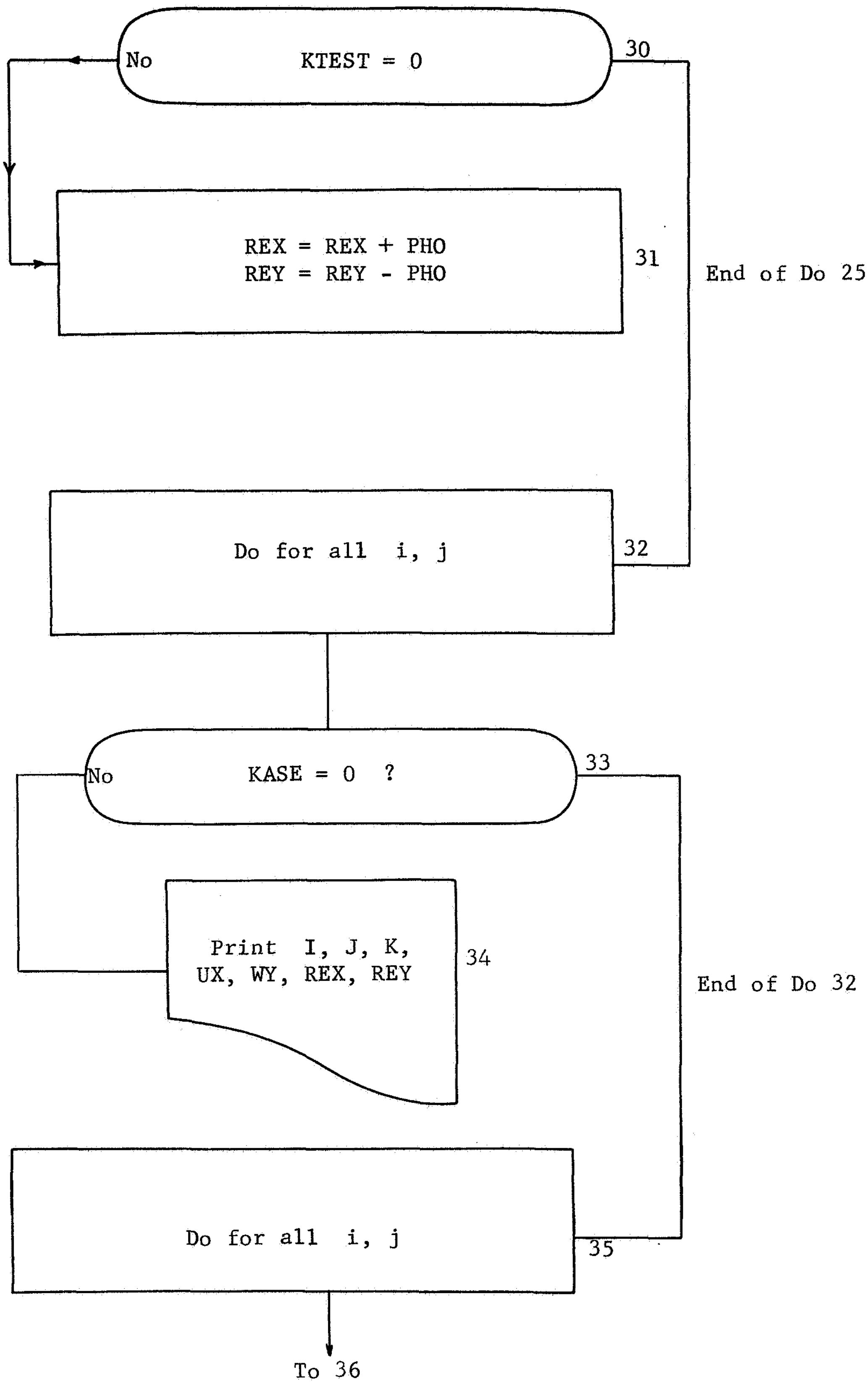


Read Specified Stresses
13
Set Specified Stresses to Zero
14
No
NTEST = 0 ?
Yes
15
Read Input Data
Uniform Displacements
16
No
ITEST = 0 ?
Yes
17
Read Input Data
Nonuniform
Displacements
18
to 19
to 20

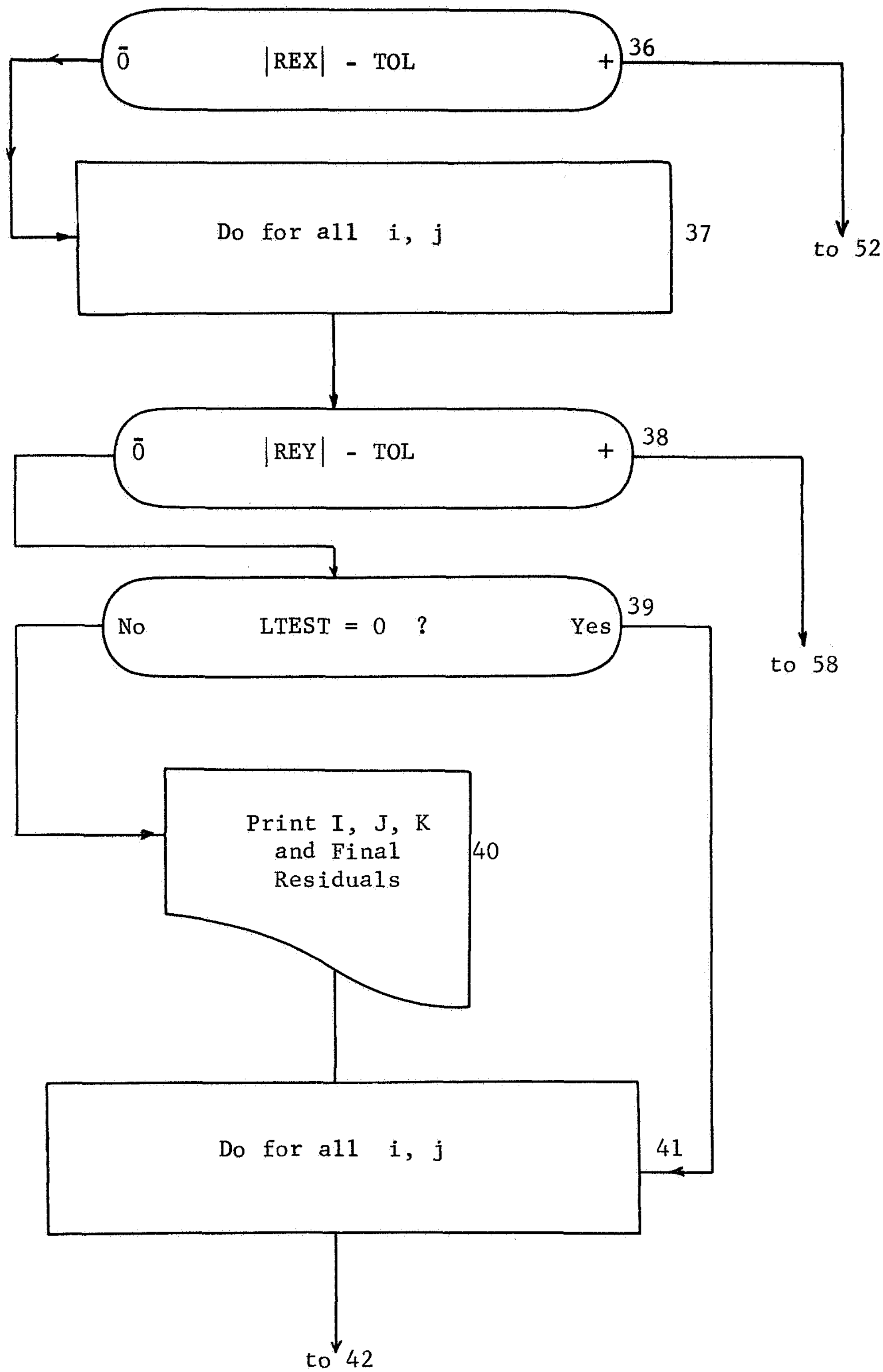


from 16
Read Input Data for K=3 only
19
from 18
Yes
KPO = 0 ?
No
20
Print Input Data
21
K = 3
22
Set Boundary Conditions
23
to 24

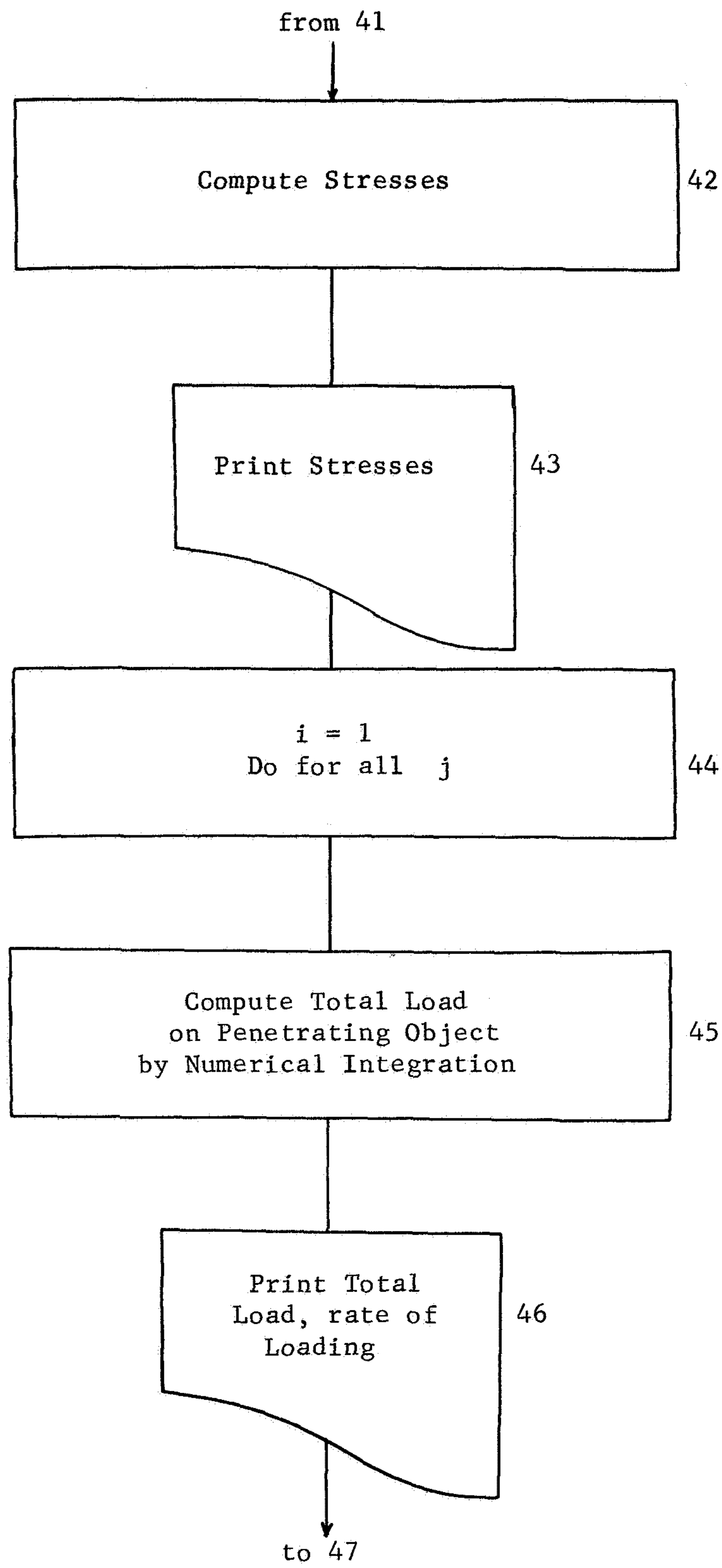

From 23
Set Initial Values of
Displacements as Equal to Those
of Previous Time Station
24
Do for all i, j
25
No
JTEST = 0
Yes
26
Read ITT1, ITT2,
JTT1, JTT2
27
Set Zero Displacements
for ITT1, ITT2, JTT1, JTT2
28
Compute REX, REY
29
to 30



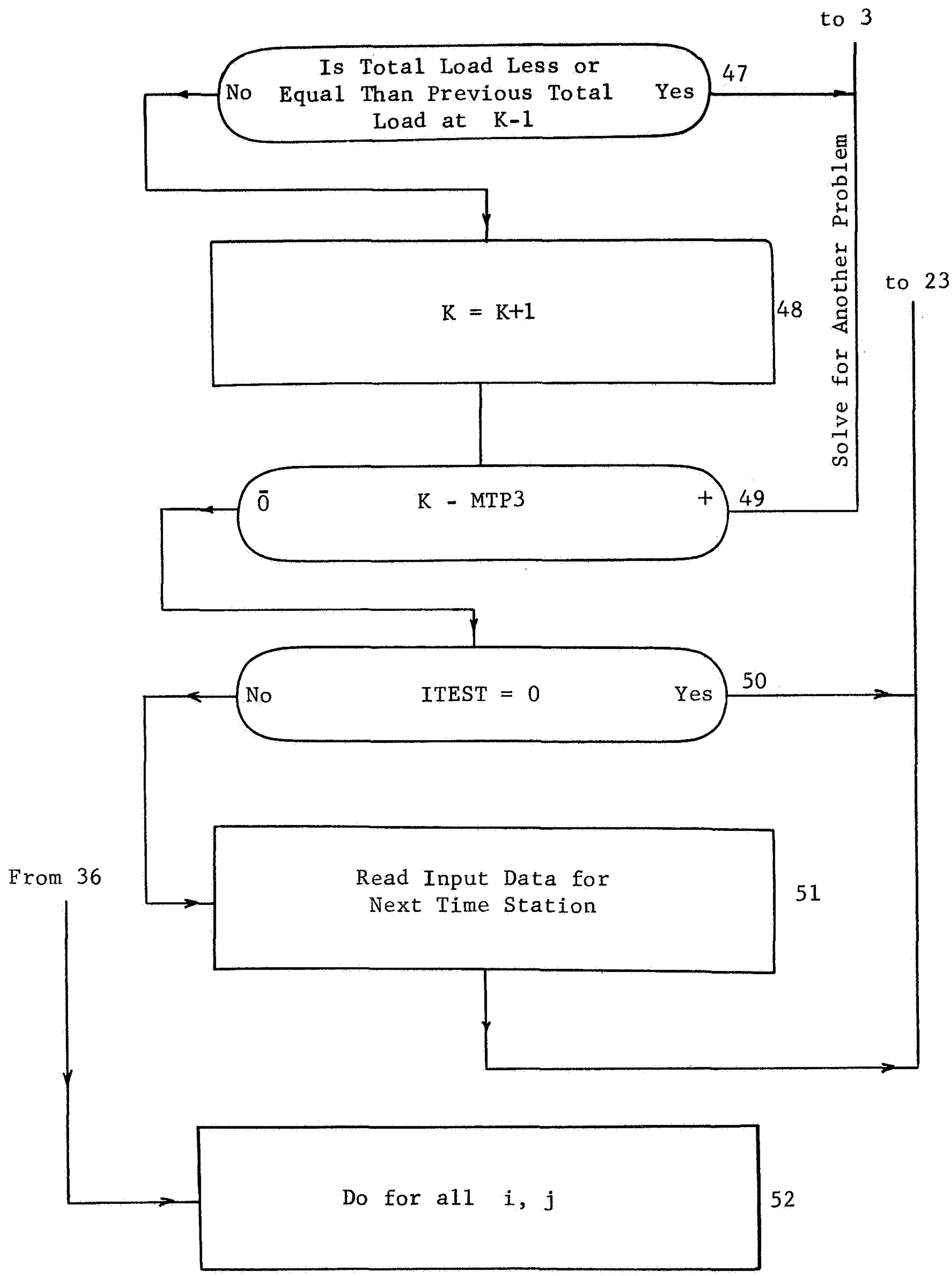
No
KTEST = 0
30
REX = REX + PHO
REY = REY - PHO
31
End of Do 25
Do for all i, j
32
No
KASE = 0 ?
33
Print I, J, K,
UX, WY, REX, REY
34
End of Do 32
Do for all i, j
35
To 36

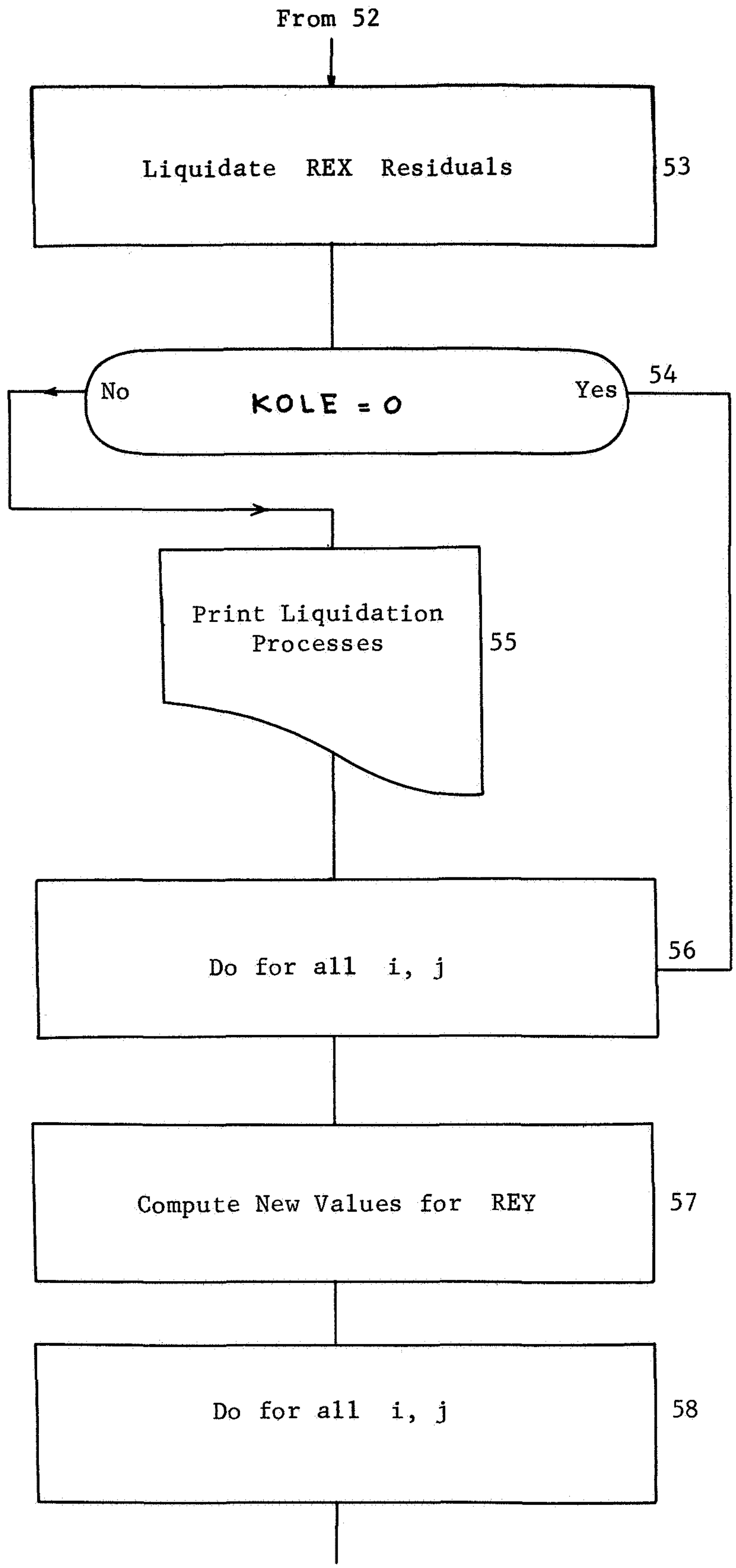


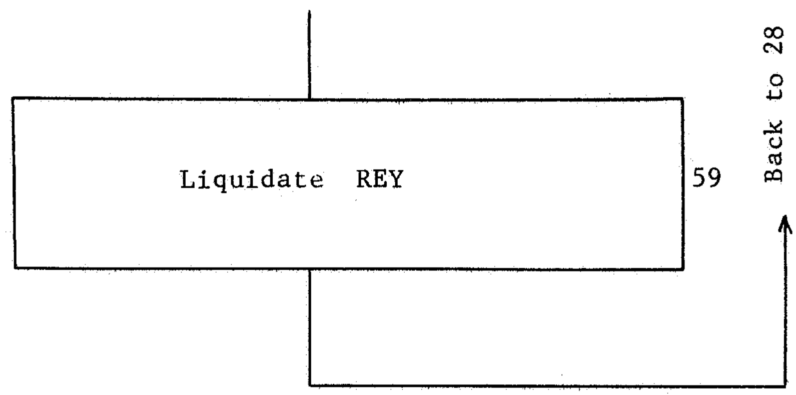
0̄
|REX| - TOL
+
36
to 52
Do for all i, j
37
0̄
|REY| - TOL
+
38
to 58
No
LTEST = 0 ?
Yes
39
Print I, J, K
and Final
Residuals
40
Do for all i, j
41
to 42



from 41
Compute Stresses
42
Print Stresses
43
i = 1
Do for all j
44
Compute Total Load
on Penetrating Object
by Numerical Integration
45
Print Total
Load, rate of
Loading
46
to 47


to 3
Is Total Load Less or Equal Than Previous Total Load at K-1
No
Yes
47
Solve for Another Problem
K = K+1
48
to 23
K - MTP3
0̄
+
49
ITEST = 0
No
Yes
50
From 36
Read Input Data for Next Time Station
51
Do for all i, j
52


From 52
Liquidate REX Residuals
53
No
KOLE = 0
Yes
54
Print Liquidation Processes
55
Do for all i, j
56
Compute New Values for REY
57
Do for all i, j
58


Liquidate REY
59
Back to 28

IDENTIFICATION OF PROBLEM AND RUN (2 ALPHANUMERIC CARDS PER RUN)

	1

IDENTIFICATION OF PROBLEM (ONE CARD, PROGRAM STOPS IF NPROB=0)

NPROB (5)	DESCRIPTION OF PROBLEM (ALPHA NUMERIC)	2

DEFORMATION COEFFICIENTS (2 CARDS)

(E10.3 format)

10	AP1	AP2	AP3	AP4	AP5	AP6	AP7	
								3
10	CP1	CP2	CP3	CP4	CP5			

CONTROL CONSTANTS (All in I5 format except TOL and TOLP in E10.3 format)

KPO	KASE	KOLE	KTEST	JTEST	ITEST	NTEST	LTEST	IM	JB	CB1	CB2	KB	LB

4

TOL	TOLP

INPUT DATA CONSTANTS, MY is not constant for the case of wedge or cylinder (that is, ITEST is different from zero), new value should be read at each time station. The value for the first time station is read in the same format as that for ITEST=0 which is the case as the following:

All in E10.3 format except MX, MY, MT in I5 format.

MX	MY	MT	ALX	ALY	HT	RHO	PHO	RATE

5

5

WD

5

SPECIFIED NORMAL STRESS ON SURFACE (HORIZONTAL SURFACE FOR JB = 1, KTEST=0, VERTICAL SURFACE FOR JB = 1, KTEST OTHER THAN ZERO). Number of cards = (MX+1) each carrying the value of stress in PSI on each station node. No input needed if CB1 and CB2 are nonzero. Such stresses will be considered zeros. Input in E10.3 format.

STY	6

INPUT DATA IF ITEST IS OTHER THAN ZERO, (that is a problem of wedge or cylinder), the number of increments MMY is read, defining the equal distances from the edges of the surface of contact to the boundaries, in the Y direction. MY and MMY change at each time station (see note for inputs) and therefore new values should be read after the solution is done for the previous time station. (That is for the first time station K=3). New values of MMY and MY will be read after the solution is obtained for K=3. At this state number of cards needed is one. No input is needed if ITEST is zero (that is a plate problem). Input in E10.3 format.

MMY	7

Input data if JB is 3, ITEST is other than zero, see note for input 7, no input is needed if JB is other than 3 or if ITEST=0. Input in E10.3 format.

MMY 5 8

Input if JB=3, ITEST is zero, YM1 is read which defines the distance from one edge of the plate to the surface where stresses are specified or set to zero (horizontal surface if KTEST is zero, vertical surface if FTEST is nonzero), no input needed if KTEST is other than three or ITEST is nonzero. Input in I5 format.

YM1 9

SPECIFIED NORMAL STRESSES ON SURFACE (when JB=3, CB2 is zero, ITEST is zero or nonzero) (see note for input 6). No input is needed if CB2 and CB1 are nonzero, stresses set to zeros. Input in E10.3 format.

STY 10

Input data, displacements at the boundary, NTEST=0, ITEST=0 (that is equal displacements of the boundary at a particular time station for a plate problem. One set of displacements are read, the program then equalizes the displacements at other nodes on the boundary to that set. Four pairs (UX, WY) of displacements are inputted through one card, corresponding to four time stations. The numbering of time stations starts from 3 and ends in MT+3, so if number of time increments is 10 then values of displacements at time station 3 to 6 included are on first card, 7 to 10 included are on the second card, 11, 12 and 13 are on the third card. The information on the third card occupies 60 columns on the fortran card, the rest is left blank. No input of this form is needed if NTEST or ITEST are nonzeros where inputs 12 or 13 are employed. Input in E10.3 format.

UX_k	WY_k	UX_{k+1}	WY_{k+1}	UX_{k+2}	WY_{k+2}	UX_{k+3}	WY_{k+3}

11

Input data, displacements at the boundary, plate problem (ITEST=0, NTEST is nonzero (nonuniform displacements at the boundary). The same format as in 11 is used, the difference is that for each node on the contact surface, its corresponding set is read. Each four pairs are inputted through one card, therefore for each time station, the number of cards is equal to (MY+1) which is the number of nodes on the contact surface, another MY+1 card is needed for the next time station, etc. So if MT=10, MY+1=6, and JSS=1, then the first card will carry four pairs of displacements describing node JSS (-1 for problems 1 and 2, and MY+1 for problems 3 and 4, YM1+1 for problems 5 and 6). The second card carries the same number of pairs describing node JSS+1, therefore for description of displacement, from K to K+3 included, six cards are needed, the last one describing the node MY+1 = JSS1. Therefore the total number of cards would be 18. Input 12 shows one set, the second set will be for K+4, through K+7, the third set for K+8 through K+11. Each set has a number of cards equal to MY+1 which is the number of nodes of contact surface. No input of this form if ITEST is nonzero or if NTEST is zero. Input in E10-3 format.

$UX_{1,jss,k}$	$WY_{1,jss,k}$	$UX_{1,jss,k+1}$	$WY_{1,jss,k+1}$	$UX_{1,jss,k+2}$	$WY_{1,jss,k+2}$	$UX_{1,jss,k+3}$	$WY_{1,jss,k+3}$	12
$UX_{1,jss1,k}$	$WY_{1,jss1,k}$	$UX_{1,jss1,k+1}$	$WY_{1,jss1,k+1}$	$UX_{1,jss1,k+2}$	$WY_{1,jss1,k+2}$	$UX_{1,jss1,k+3}$	$WY_{1,jss1,k+3}$	

DATA INPUT, NONZERO ITEST (That is a cylinder or wedge problem.)

In this case, the dimension of contact surface varies at each time station (see input 8); therefore, information for each node on the contact surface is inputted after the solution is obtained for the previous time station. To start with, information is inputted only for the starting time station, K = 3. Therefore one set is needed. Number of cards will be equal to number of nodes (that is MY+1 at K = 3). Each card carries one pair of displacements (UX, WY). No input if ITEST is zero. Input in E10.3 format.

$UX_{1,iss,k}$ $WY_{1,iss,k}$

|

$UX_{1,iss1,k}$ $WY_{1,iss1,k}$

13

DATA INPUT IF ANY RIGID INCLUSION IS PRESENT (JTEST is nonzero). Stations ITT1 through ITT2 defining the dimension in X direction, stations JTT1. Through JTT2 defining the dimension in Y direction are inputted. No input if JTEST is zero. Input in I5 format.

ITT1	ITT2	JTT1	JTT2	14

DATA INPUT (ITEST is nonzero). New values of MY and MMY have to be inputted corresponding to K+1 (compare with input 13, at this stage solution is complete for station k, to obtain a solution for K+1, new MMY and MY has to be read). MY and MMY are read on one card, pairs of displacements for each node are read in form similar to input 13. This set is for K+1, other set follows for K+2. The last set will be for K = MT+3. Number of cards is equal to MY+1. No input for zero ITEST.

MY	MMY	Input in I5 Format

$UX_{1,iss,k+1}$	$WY_{iss,k+1}$	Input in E10.3 Format	15
⋮			
$UX_{1,iss1,k+1}$	$WY_{iss1,k+1}$		

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